STRONG LAWS IN FINITE MODEL THEORY

By

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Abstract

We introduce a new framework for asymptotic probabilities of sentences, in which we have a σ-additive measure on the sample space of all sequences $\mathbf{A} = \{\mathcal{A}_n\}$ of finite models, where the universe of $\mathcal{A}_n$ is {1,2,..,n}, and use this framework to strengthen 0-1 laws for logics and to formulate a stronger version of convergence of formulas.

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Chapter 1

Introduction.

One of the nicest theorems of Finite Model Theory is the 0-1 law of the first order logic, proved by Glebskii et. al. [7] and independently by Fagin [6]. This theorem states that the fraction of the finite relational models of fixed size that satisfy some first order sentence asymptotically tends to 0 or 1 as the size of the models tends to infinity.

This 0-1 law has been extended in several directions. First, the first order logic was replaced by more powerful logics like the infinitary logic $\mathcal{L}^{\omega}_{\infty\omega}$ with only finitely many variables.

Second, the finite models can be restricted to be in some preassigned class like the class of equivalence relations or the class of partial orders.

The word "fraction" in the statement of the 0-1 law indicates a uniform measure on the finite models of fixed size. So changing the measure is still another direction of extending the 0-1 law. In fact, restricting the finite models to some class can be seen as having a measure which is uniform on that class and 0 elsewhere.

However, a common problem of the 0-1 law and its extensions is that the "measure" μ on sentences defined by $\mu(\phi) = \lim_n \mu_n(\phi)$ (where μ_n is the measure chosen on the set of finite models of size n) is not σ-additive. In this thesis we'll construct a new framework with a (σ-additive) measure. In Chapter 2 the basic definitions of the framework are given. The (σ-additive) measure is defined on the set of all increasing sequences of finite models, rather than the set of finite models itself.

Through this framework a 0-1 law can be formulated (Chapter 3), which will turn out to be stronger than the regular 0-1 law.

Chapter 4 is devoted to finding measures on the set of finite models. The strong 0-1 law plays an important role here.

In Chapter 5 we develop a parallel theory to that of almost everywhere equivalence and show that the strong 0-1 law holds in many cases where a 0-1 law holds.

In Chapter 6 we start by getting some easy strong 0-1 laws from the literature. We then proceed to get some nontrivial results, in which we strengthen old proofs of the 0-1 law as well as find some counterexamples.

Chapter 7 then discusses the possibility of analogues for Baire category of the strong 0-1 law.

We also use our framework to define a new notion of strong convergence law for formulas, which strengthens the 0-1 law for sentences (Chapter 8). While developing the theory of strong convergence laws we bring into attention the notion of almost sure quantifier elimination and give a rather elegant proof of a theorem that subsumes the 0-1 law for first order logic with respect to the uniform measure. We then proceed to prove the strong convergence laws for other logics.

Chapter 9, which parallels Chapter 6, investigates the strong convergence law for nonuniform measures.

At the end, Chapter 8 concludes by some open problems.

Chapter 2

Basic definitions.

Fix a finite relational vocabulary ν, and consider only ν-models. Define $\mathbf{N} = \{1, 2, ..\}$.

For each $n \in \mathbf{N}$, let $\mathbf{M}_n = \{\mathcal{A} : \mathcal{A}$ is a ν-model with universe $|\mathcal{A}| = \{1, .., n\}\}$, and let $\mathbf{M} = \bigcup_{n\in\mathbf{N}} \mathbf{M}_n$.

Also let $\mathcal{F}_n = \mathcal{P}(\mathbf{M}_n)$ be the (σ-)algebra of all subsets of $\mathbf{M}_n$, and let μ_n be some probability measure on $\mathcal{F}_n$.

Our sample space will be the Cartesian product of the $\mathbf{M}_n$s, i.e. $\Omega = \prod_{n\in\mathbf{N}} \mathbf{M}_n$, with a typical element $\mathbf{A} = \langle \mathcal{A}_n \rangle_{n\in\mathbf{N}}$ being a sequence of models.

Define the projection on the n-th coordinate $\pi_n : \Omega \to \mathbf{M}_n$ by $\pi_n(\mathbf{A}) = \mathcal{A}_n$. Now let $\mathcal{F}$ be the σ-algebra on Ω generated by the cylindrical events $\pi_n^{-1}(E)$ for $E \subseteq \mathbf{M}_n$, i.e. $\mathcal{F}$ is the least σ-algebra that makes each π_n measurable.

Identifying the (σ-)algebras $\mathcal{F}_n$ with the cylindrical sub-σ-algebras,

$$\{\mathbf{M}_1\} \times .. \times \{\mathbf{M}_{n-1}\} \times \mathcal{F}_n \times \{\mathbf{M}_{n+1}\} \times ..,$$

of $\mathcal{F}$, we let μ be the σ-additive product probability measure on $\mathcal{F}$ satisfying $\mu(E) = \mu_n(E)$ for every $E \in \mathcal{F}_n$. We call this measure μ a product measure on Ω, and denote its probability space by

$$(\Omega, \mathcal{F}, \mu) = \prod_{n\in\mathbf{N}} (\mathbf{M}_n, \mathcal{F}_n, \mu_n).$$

The existence and uniqueness of μ is guaranteed by a well known theorem of Kolmogorov's (see e.g. [8]). Throughout this thesis μ is understood to be a product measure on Ω.

Note that the product probability space $(\Omega, \mathcal{F}, \mu)$, which depends on the underlying vocabulary ν, represents the experiment of choosing models $\mathcal{A}_n$ with universe $\{1, .., n\}$, where the $\mathcal{A}_n$ are chosen independently of each other according to the measures μ_n.

Example: Let $\mathbf{C} \subseteq \mathbf{M}$ be a class of models that contains models of each size $n \in \mathbf{N}$. Taking μ_n to be the uniform measure on $\mathbf{C} \cap \mathbf{M}_n$ and 0 elsewhere, we get a product probability measure $\mu = \prod_{n \in \mathbf{N}} \mu_n$ called the uniform measure on $\mathbf{C}$. The uniform measure is then defined to be the uniform measure on $\mathbf{M}$.

Note that if the spectrum of the class $\mathbf{C}$, defined by $Spectrum(\mathbf{C}) = \{n \in \mathbf{N} : \mathbf{C} \cap \mathbf{M}_n \neq \emptyset\}$, is not the whole set $\mathbf{N}$ but an infinite subset thereof, then the uniform measure (or any other measure supported) on $\mathbf{C}$ can be defined as above but replacing each $\mathbf{N}$ in the definition of our sample space by the set $Spectrum(\mathbf{C})$.

For a class $\mathbf{C} \subseteq \mathbf{M}$, let

$$\mathcal{C}^n = \{\mathbf{A} \in \Omega : \mathcal{A}_n \in \mathbf{C}\},$$

thus $\mathcal{C}^n = \pi_n^{-1}(\mathbf{C})$ and $\mu(\mathcal{C}^n) = \mu_n(\mathbf{C} \cap \mathbf{M}_n)$.

In the literature $\mu_n(\mathbf{C} \cap \mathbf{M}_n)$ is denoted by $\mu_n(\mathbf{C})$, but in our framework we concentrate on the measure μ shifting the subscript n in μ_n to a superscript in $\mathcal{C}^n$, and we'll be using both $\mu_n(\mathbf{C})$ and $\mu(\mathcal{C}^n)$ interchangeably.

Define

$$\underline{\mathbf{C}} = \underline{\lim}_n \, \mathcal{C}^n = \bigcup_N \bigcap_{n \geq N} \mathcal{C}^n,$$

and

$$\overline{\mathbf{C}} = \overline{\lim}_n \, \mathcal{C}^n = \bigcap_N \bigcup_{n \geq N} \mathcal{C}^n.$$

Thus $\mathbf{A} \in \underline{\mathbf{C}}$ iff ($\mathcal{A}_n \in \mathbf{C}$ eventually(n)),

and $\mathbf{A} \in \overline{\mathbf{C}}$ iff ($\mathcal{A}_n \in \mathbf{C}$ infinitely often(n)).

Clearly $\underline{\mathbf{M} \setminus \mathbf{C}} = \mathbf{M} \setminus \overline{\mathbf{C}}$, thus we'll only need to deal with one of them.

To investigate the convergence of $\mu_n(\mathbf{C}) = \mu(\mathcal{C}^n)$, we look at the liminf and the limsup of $\mu(\mathcal{C}^n)$. We easily get:

Proposition 2.1

$$\mu(\underline{\mathbf{C}}) \leq \underline{\lim}_n \, \mu(\mathcal{C}^n) \leq \overline{\lim}_n \, \mu(\mathcal{C}^n) \leq \mu(\overline{\mathbf{C}}),$$

where the first and the third inequalities are justified by Fatou's lemma for events.

∎

However, the extreme sides of the inequalities can only take trivial values, as shown in the next proposition.

Proposition 2.2 *One of the following 3 cases holds:*

(1) $\mu(\underline{\mathbf{C}}) = \mu(\overline{\mathbf{C}}) = 0$.

(2) $\mu(\underline{\mathbf{C}}) = \mu(\overline{\mathbf{C}}) = 1$.

(3) $\mu(\underline{\mathbf{C}}) = 0$ *and* $\mu(\overline{\mathbf{C}}) = 1$.

Proof: Note that the events $\underline{\mathbf{C}}$ and $\overline{\mathbf{C}}$ are tail events of the independent σ-algebras $\mathcal{F}_n$, i.e. $\underline{\mathbf{C}}, \overline{\mathbf{C}} \in \mathcal{T}$, where $\mathcal{T} = \bigcap_N \sigma\{\mathcal{F}_n : n \geq N\}$. Therefore by

Kolmogorov's 0-1 Law they have μ-probabilities 0 or 1. The result then follows from Proposition 2.1. ∎

Also, since $\mathcal{C}^n$ are independent, by the first and second Borel-Cantelli lemmas we get:

Proposition 2.3

$$\mu(\overline{\mathbf{C}}) = 0 \iff \sum_n \mu_n(\mathbf{C}) < \infty.$$

∎

Example: Let the vocabulary ν contain a unary relation symbol R and let $\mathbf{C} = \{\mathcal{A} : (\mathcal{A}$ has odd size and either 1 or $2 \in R^{\mathcal{A}})$ or $(\mathcal{A}$ has even size and both $1, 2 \in R^{\mathcal{A}})\}$. Then the inequality in Proposition 2.1 reads: $0 \leq \frac{1}{4} \leq \frac{3}{4} \leq 1$.

Chapter 3

The strong 0-1 law.

Next we identify a sentence ϕ with $Mod(\phi) = \{\mathcal{A} \in \mathbf{M} : \mathcal{A} \models \phi\}$, and a logic $\mathcal{L}$ with the set of formulas in $\mathcal{L}$.

Recall that in the literature a logic $\mathcal{L}$ is said to have the 0-1 law for μ if for every sentence ϕ in $\mathcal{L}$, $\lim_n \mu_n(\phi) = 0$ or 1. Now in view of Proposition 2.2 we have the following definition.

Definition 3.1 *A logic $\mathcal{L}$ has the strong 0-1 law for a product measure μ if for every sentence $\phi \in \mathcal{L}$,*

either $\mu(\underline{\phi}) = 1$ or $\mu(\overline{\phi}) = 0$,

(equivalently $\mu(\underline{\phi}) = \mu(\overline{\phi})$).

In other words either $\mathcal{A}_n \models \phi$ eventually almost surely,

or $\mathcal{A}_n \models \neg\phi$ eventually almost surely.

In view of proposition 2.1 we get:

Corollary 3.2 *A logic that has the strong 0-1 law for μ also has the 0-1 law for μ.*

■

Also we see:

Corollary 3.3 *Let μ, μ' be two product measures which agree on the tail σ-algebra $\mathcal{T}$ (in particular we may have $\mu \ll \mu'$ or $\mu' \ll \mu$). Then a logic that has the strong 0-1 law for μ also has the strong 0-1 law for μ'.*

∎

Note that for two product measures μ and μ', if $\mu \ll \mu'$ (i.e. μ is absolutely continuous with respect to μ'), then they have to agree on the tail σ-algebra $\mathcal{T}$, since μ' is trivial on $\mathcal{T}$.

Remark: Consider the independent real random variables $X_1, X_2, ..$ with 0 expectation, and with the sum random variables being defined as $S_n = \sum_{k=1}^{n} X_k$. Recall that the Weak Law of Large Numbers can be stated as:

For every $\epsilon > 0$, $\lim_n P(|S_n/n| > \epsilon) = 0$.

While the Strong Law of Large Numbers can be stated as:

For every $\epsilon > 0$, $P(|S_n/n| > \epsilon \ \ i.o.) = 0$.

Replacing P with our product measure μ and the events $|S_n/n| > \epsilon$ with the cylindrical events ϕ^n, we see that the Strong 0-1 Law defined above has the same relationship to the 0-1 Law that the Strong Law of Large Numbers has to the Weak Law of large numbers. This answers a question posed my Mycielski.

Fagin's proof of the 0-1 law of first order logic in [6] proves:

Theorem 3.4 *First order logic has the strong 0-1 law for the uniform measure.*

Proof: As in [6] we consider the ω-categorical (and hence complete) random theory Ψ of all extension axioms. From completeness of Ψ we get:

For every first order sentence ϕ, either $\Psi \models \phi$ or $\Psi \models \neg\phi$.

Assume the latter, then from compactness there is a finite subset $\Psi_0 \subset \Psi$ such that: $\Psi_0 \models \neg\phi$.

Thus $\phi \models \bigvee_{\psi\in\Psi_0}(\neg\psi)$, so $\mu_n(\phi) \leq \sum_{\psi\in\Psi_0}\mu_n(\neg\psi)$.

Now from Fagin's proof we know that for each extension axiom ψ:

$$\mu_n(\neg\psi) \leq n^m \left(1-(1/2)^{2m+1}\right)^{n-m},$$

where m denotes the number of variables in ψ.

Thus we get:

$$\sum_n \mu_n(\phi) \leq \sum_{\psi\in\Psi_0}\sum_n \mu_n(\neg\psi) \leq \sum_{\psi\in\Psi_0}\sum_n n^m \left(1-(1/2)^{2m+1}\right)^{n-m} < \infty,$$

and from Proposition 2.3 we get that $\mu(\overline{\phi}) = 0$.

In the case of $\Psi \models \phi$, we similarly get $\mu(\overline{\neg\phi}) = 0$, and hence $\mu(\underline{\phi}) = 1$. ∎

Both Fagin [6] and Glebskii et. al. [7] noted that for a first order sentence ϕ, the sequence $\mu_n(\phi)$ converges exponentially fast to either 0 or 1, which clearly entails the strong 0-1 law. However, our formalization gives a new meaning of that (fast) tendency of the sequence, as well as a new method to deal with asymptotic probabilities.

Chapter 4

Finding measures on M.

If for some sentence ϕ we have $\mu(\underline{\phi}) = \mu(\overline{\phi})$, then we may define $\hat{\mu}(\phi)$ to be that common value. Thus if a logic $\mathcal{L}$ has the strong 0-1 law for the product measure μ, $\hat{\mu}(\phi)$ will be defined for every sentence $\phi \in \mathcal{L}$.

We note that μ is a (σ-additive) measure on the sample space Ω, but in general $\hat{\mu}$ is not necessarily σ-additive on the set of finite models $\mathbf{M}$, even if it is always defined.

For example if $\mathcal{L} = \mathbf{FO}$, then for any measure μ we have that for each model $\mathcal{A} \in \mathbf{M}$, $\hat{\mu}(\{\mathcal{A}\}) = 0$, but of course

$$\hat{\mu}\left(\bigcup_{\mathcal{A}\in\mathbf{M}} \{\mathcal{A}\}\right) = \hat{\mu}(\mathbf{M}) = 1.$$

So (since $\mathbf{M}$ is countable) $\hat{\mu}$ is not σ-additive.

Actually if each singleton in $\mathbf{M}$ is measurable, we have no hope of getting a σ-additive (asymptotic) measure on $\mathbf{M}$.

However, the problem with $\mathbf{FO}$ is more profound since $(\mathbf{M}, \mathbf{FO})$ is not even a measurable space, as $\mathbf{FO}$ is not a σ-algebra. (Note that here $\mathbf{FO}$ is viewed as the set of sentences rather than formulas.)

The following proposition shows that once we have a measurable space, the measure $\hat{\mu}$ (if defined) is σ-additive as required. The equivalence between (1) and

(2) in the proposition parallels Theorem 3.15 in [16] with an easier proof.

Definition 4.1 *A sentence ψ in a logic $\mathcal{L}$ is said to be minimal in $\mathcal{L}$ iff for every sentence $\phi \in \mathcal{L}$, either $\phi \cap \psi = \emptyset$ or $\phi \cap \psi = \psi$.*

Proposition 4.2 *Let $\mathcal{L}$ be a logic, for which the set of sentences (up to equivalence) is closed under negation, countable conjunction and disjunction. (In other words $\mathcal{L}$ is a σ-algebra on $\mathbf{M}$, so $(\mathbf{M}, \mathcal{L})$ is a measurable space.) Then the following statements are equivalent:*

(1) $\mathcal{L}$ has the strong 0-1 law for μ.

(2) There is a sentence $\psi \in \mathcal{L}$ such that $\hat{\mu}(\psi) = 1$ and ψ is minimal in $\mathcal{L}$.

(3) $(\mathbf{M}, \mathcal{L}, \hat{\mu})$ is a 0-1 measure space.

Note that the logic $\mathcal{L}$ can be typically taken to be $\mathcal{L}^k_{\infty\omega}$ (the infinitary logic with k many variables, see e.g. [16]) or $\mathbf{FO}_r$ (first order logic with quantifier rank $\leq r$, which has only finitely many nonequivalent sentences).

Proof: $(3 \Rightarrow 1)$ is clear.

$(1 \Rightarrow 2)$: Assume every minimal sentence $\psi \in \mathcal{L}$ has $\hat{\mu}(\psi) = 0$. Since $\mathbf{M}$ is countable, let $\{\psi_i\}_{i \in \mathbf{N}}$ be an enumeration of the minimal ψs. Since for every $i \in \mathbf{N}$, $\hat{\mu}(\psi_i) = 0$, we know that

$$K_i := \sum_n \mu_n(\psi_i) < \infty.$$

However,

$$\sum_i K_i = \sum_i \sum_n \mu_n(\psi_i) = \sum_n \sum_i \mu_n(\psi_i) = \sum_n \mu_n\left(\bigcup_i \psi_i\right) = \sum_n \mu_n(\mathbf{M}) = \infty.$$

So we can easily find a set $I \subset \mathbf{N}$ with both $\sum_{i \in I} K_i = \sum_{i \notin I} K_i = \infty$.

Now let $\phi \in \mathcal{L}$ be a sentence equivalent to $\bigvee_{i \in I} \psi_i$. Then

$$\sum_n \mu_n(\phi) = \sum_n \mu_n \left(\bigvee_{i \in I} \psi_i \right) = \sum_n \sum_{i \in I} \mu_n(\psi_i) = \sum_{i \in I} \sum_n \mu_n(\psi_i) = \sum_{i \in I} K_i = \infty.$$

Thus, $\mu(\overline{\phi}) = 1$. Similarly:

$$\sum_n \mu_n(\neg\phi) = \sum_n \mu_n \left(\bigvee_{i \notin I} \psi_i \right) = \sum_{i \notin I} K_i = \infty.$$

So $\mu(\overline{\neg\phi}) = 1$, i.e. $\mu(\underline{\phi}) = 0$, contradicting (1).

$(2 \Rightarrow 3)$: Since ψ is minimal, for every sentence $\phi \in \mathcal{L}$,

either $\phi \cap \psi = \emptyset$, i.e. $\phi \subseteq \neg\psi$, in which case

$$\mu(\overline{\phi}) \leq \mu(\overline{\neg\psi}) = 1 - \mu(\underline{\psi}) = 1 - 1 = 0,$$

or $\phi \cap \psi = \psi$, i.e. $\phi \supseteq \psi$, in which case $\mu(\underline{\phi}) \geq \mu(\underline{\psi}) = 1$.

Thus, $\hat{\mu}(\phi)$ is defined for each $\phi \in \mathcal{L}$.

For σ-additivity let $\phi_n \in \mathcal{L}$ be disjoint sentences, with $\hat{\mu}(\phi_n) = 0$. From the minimality of ψ we must have $\phi_n \cap \psi = \emptyset$ for every n. Then

$$\hat{\mu}\left(\bigcup_n \phi_n \right) = \hat{\mu}\left(\left(\bigcup_n \phi_n \right) \cap \psi \right) = \hat{\mu}\left(\bigcup_n (\phi_n \cap \psi) \right) = \hat{\mu}\left(\bigcup_n \emptyset \right) = \hat{\mu}(\emptyset) = 0.$$

Thus $\hat{\mu}$ is a σ-additive 0-1 measure defined on $\mathcal{L}$. ∎

Note that if $\hat{\mu}$ is σ-additive then it will be even uncountably additive, since there are only countably many nonempty disjoint subsets of $\mathbf{M}$.

Chapter 5

Almost sure equivalence.

We first include the definition of the notion of almost everywhere reduction of one logic to another, defined in [10].

Definition 5.1 *For two logics $\mathcal{L}$ and $\mathcal{L}'$, we say that $\mathcal{L}$ is less than or equal to $\mathcal{L}'$ almost everywhere (with respect to μ) and write $\mathcal{L} \leq_{a.e.} \mathcal{L}'(\mu)$ if:*

There is a class of models $\mathbf{C} \subseteq \mathbf{M}$ with $\lim_n \mu_n(\mathbf{C}) = 1$ such that,

for every $\phi(\mathbf{x}) \in \mathcal{L}$, there is a $\phi'(\mathbf{x}) \in \mathcal{L}'$, with

$$\mathbf{C} \subseteq \forall \mathbf{x}(\phi(\mathbf{x}) \leftrightarrow \phi'(\mathbf{x})).$$

We say that $\mathcal{L}$ is less than or equal to $\mathcal{L}'$ weakly almost everywhere (w.r.t. μ) and write $\mathcal{L} \leq_{w.a.e.} \mathcal{L}'(\mu)$ if:

For every $\phi(\mathbf{x}) \in \mathcal{L}$, there is a $\phi'(\mathbf{x}) \in \mathcal{L}'$, with

$$\lim_n \mu_n(\forall \mathbf{x}(\phi(\mathbf{x}) \leftrightarrow \phi'(\mathbf{x}))) = 1.$$

The notions $\mathcal{L} \leq_{a.e.} \mathcal{L}'(\mu)$ and $\mathcal{L} \leq_{w.a.e.} \mathcal{L}'(\mu)$ are easily seen to be preorders on the class of logics. We denote by $\mathcal{L} \equiv_{a.e.} \mathcal{L}'$ and $\mathcal{L} \equiv_{w.a.e.} \mathcal{L}'$ the equivalence relations they induce.

Now we introduce our new definition that strengthens the previous one.

Definition 5.2 *For two logics $\mathcal{L}$ and $\mathcal{L}'$, we define the notions of:*

$\mathcal{L} \leq_{a.s.} \mathcal{L}'(\mu)$ *($\mathcal{L}$ is less than or equal to $\mathcal{L}'$ almost surely (with respect to μ)),*

and

$\mathcal{L} \leq_{w.a.s.} \mathcal{L}'(\mu)$ *($\mathcal{L}$ is less than or equal to $\mathcal{L}'$ weakly almost surely (with respect to μ))*

exactly as in the definitions of $\mathcal{L} \leq_{a.e.} \mathcal{L}'(\mu)$, and $\mathcal{L} \leq_{w.a.e.} \mathcal{L}'(\mu)$, but replacing $\lim_n \mu_n(\mathbf{C}) = 1$ by $\mu(\underline{\mathbf{C}}) = 1$ in the first definition, and $\lim_n \mu_n(\forall \mathbf{x}(\phi(\mathbf{x}) \leftrightarrow \phi'(\mathbf{x}))) = 1$ by $\mu(\underline{\forall \mathbf{x}(\phi(\mathbf{x}) \leftrightarrow \phi'(\mathbf{x}))}) = 1$ in the second.

As in Definition 5.1, $\mathcal{L} \leq_{a.s.} \mathcal{L}'(\mu)$ and $\mathcal{L} \leq_{w.a.s.} \mathcal{L}'(\mu)$ are preorders on the class of logics, and we write $\mathcal{L} \equiv_{a.s.} \mathcal{L}'$ and $\mathcal{L} \equiv_{w.a.s.} \mathcal{L}'$ for the equivalence relations they induce.

From the definitions we immediately get:

$$\mathcal{L} \leq_{a.s.} \mathcal{L}'(\mu) \implies \mathcal{L} \leq_{w.a.s.} \mathcal{L}'(\mu) \implies \mathcal{L} \leq_{w.a.e.} \mathcal{L}'(\mu),$$

and

$$\mathcal{L} \leq_{a.s.} \mathcal{L}'(\mu) \implies \mathcal{L} \leq_{a.e.} \mathcal{L}'(\mu) \implies \mathcal{L} \leq_{w.a.e.} \mathcal{L}'(\mu).$$

As in the case of almost everywhere equivalence, we can sometimes reverse the first implication. We prove first a lemma similar to Lemma 3.3 in [10].

Lemma 5.3 *Assume that for each $k \in \mathbf{N}$ we have a class $\mathbf{C}_k$ of models such that $\mu(\underline{\mathbf{C}_k}) = 1$. Then there exists a class $\mathbf{C}$ of models such that $\mu(\underline{\mathbf{C}}) = 1$ and for every $k \in \mathbf{N}$ the set difference $\mathbf{C} \setminus \mathbf{C}_k$ is finite (and hence $\underline{\mathbf{C}} \subseteq \underline{\mathbf{C}_k}$).*

Remark: The lemma states that $\bigcap_k \underline{\mathbf{C}_k}$ (which has measure 1 by σ-additivity) contains some set $\underline{\mathbf{C}}$ of measure 1.

Proof: Since $\mu(\underline{\mathbf{C}_k}) = 1$, from Proposition 2.3 we get that

$\sum_n (1 - \mu_n(\mathbf{C}_k)) < \infty$. Choose a strictly increasing function $f : \mathbf{N} \longrightarrow \mathbf{N}$ such that

$$\sum_{n \geq f(k)} (1 - \mu_n(\mathbf{C}_k)) \leq \frac{1}{2^k},$$

and define $f^{-1}(n) = \max\{k : f(k) \leq n\}$, so that we get:

$$k \leq f^{-1}(n) \iff f(k) \leq n,$$

and $\lim_n f^{-1}(n) = \infty$. Now define:

$$\mathbf{C} = \bigcup_n \left(\bigcap_{k \leq f^{-1}(n)} \mathbf{C}_k \cap \mathbf{M}_n \right).$$

Then we get:

$$\begin{aligned}
\sum_n (1 - \mu_n(\mathbf{C})) &= \sum_n \left(1 - \mu_n \left(\bigcap_{k \leq f^{-1}(n)} \mathbf{C}_k \right)\right) \\
&= \sum_n \mu_n \left(\bigcup_{k \leq f^{-1}(n)} (\mathbf{M} \setminus \mathbf{C}_k) \right) \\
&\leq \sum_n \sum_{k \leq f^{-1}(n)} \mu_n(\mathbf{M} \setminus \mathbf{C}_k) \\
&= \sum_k \sum_{n \geq f(k)} (1 - \mu_n(\mathbf{C}_k)) \\
&\leq \sum_k \frac{1}{2^k} \\
&< \infty.
\end{aligned}$$

Thus, by Proposition 2.3 again, $\mu(\underline{\mathbf{C}}) = 1$, and since for every k,

$$\mathbf{C} \subseteq \mathbf{C}_k \cup \bigcup_{n < f(k)} \mathbf{M}_n,$$

we get that $\mathbf{C} \setminus \mathbf{C}_k \subseteq \bigcup_{n<f(k)} \mathbf{M}_n$, which is finite. ∎

The next theorem parallels Theorem 3.4. in [10].

Theorem 5.4 *Let $\mathcal{L}$, $\mathcal{L}'$ be two logics such that $\mathcal{L}$ is countable, and $\mathcal{L}'$ has the following closure property:*

For every $\phi'(\mathbf{x}) \in \mathcal{L}'$ and $\tilde{\phi}(\mathbf{x}) \in \mathcal{L}_{\infty\omega}$ such that $\neg\forall\mathbf{x}(\phi'(\mathbf{x}) \leftrightarrow \tilde{\phi}(\mathbf{x}))$ is finite (i.e. has finitely many models), we also have $\tilde{\phi}(\mathbf{x}) \in \mathcal{L}'$.

Then

$$\mathcal{L} \leq_{a.s.} \mathcal{L}'(\mu) \iff \mathcal{L} \leq_{w.a.s.} \mathcal{L}'(\mu).$$

Note that if $\mathcal{L}' \supseteq \mathbf{FO}$ and is closed under Boolean connectives, then $\mathcal{L}'$ has the required closure property.

Proof: For each $\phi(\mathbf{x}) \in \mathcal{L}$ let $\phi'(\mathbf{x}) \in \mathcal{L}'$ be such that

$$\mu(\underline{\forall\mathbf{x}(\phi(\mathbf{x}) \leftrightarrow \phi'(\mathbf{x}))}) = 1.$$

Since $\mathcal{L}$ is countable we can use Lemma 5.3 to get $\mathbf{C} \subseteq \mathbf{M}$ such that $\mu(\underline{\mathbf{C}}) = 1$ and for every $\phi(\mathbf{x}) \in \mathcal{L}$ the set difference $\mathbf{C} \setminus \forall\mathbf{x}(\phi(\mathbf{x}) \leftrightarrow \phi'(\mathbf{x}))$ is finite. Now using the closure property of $\mathcal{L}'$, for every $\phi(\mathbf{x}) \in \mathcal{L}$ get a formula $\tilde{\phi}(\mathbf{x}) \in \mathcal{L}'$ such that

$$\mathbf{C} \subseteq \forall\mathbf{x}(\phi(\mathbf{x}) \leftrightarrow \tilde{\phi}(\mathbf{x})).$$

∎

Remark:

-If we replace the inclusion $\mathbf{C} \subseteq \forall\mathbf{x}(\phi(\mathbf{x}) \leftrightarrow \phi'(\mathbf{x}))$ (in the def of $\mathcal{L} \leq_{a.s.} \mathcal{L}'$) by $\underline{\mathbf{C}} \subseteq \underline{\forall\mathbf{x}(\phi(\mathbf{x}) \leftrightarrow \phi'(\mathbf{x}))}$, then we get a weaker notion of $\leq_{a.s.}$, unless $\mathcal{L}'$ has the closure property of the last theorem. In this case the two notions coincide.

Note that if $\mathbf{C}$ has a cofinite spectrum, i.e. $\mathbf{C} \cap \mathbf{M}_n \neq \emptyset$ for sufficiently large n, then:

$\underline{\mathbf{C}} \subseteq \underline{\mathbf{C}'}$ iff $\mathbf{C} \setminus \mathbf{C}'$ is finite.

-If we further replace $\underline{\mathbf{C}}$ in the def of $\mathcal{L} \leq_{a.s.} \mathcal{L}'$ by an arbitrary event $\mathcal{E}$ of measure 1, and require that $\mathcal{E} \subseteq \underline{\forall \mathbf{x}(\phi(\mathbf{x}) \leftrightarrow \phi'(\mathbf{x}))}$, then we get a still weaker notion of $\leq_{a.s.}$ (but stronger than $\leq_{w.a.s.}$). The weakness of that notion results from the fact that there is an event $\mathcal{E}$ of measure 1 that does not include any event of the form $\underline{\mathbf{C}}$ for some class of models $\mathbf{C}$ of measure bigger than 1/2.

For an example of such an event $\mathcal{E}$, let the vocabulary ν contain a relation symbol U, and take

$$\mathcal{E} = \{\mathbf{A} : \lim_n \frac{\sum_{k \leq n} U_k(1)}{n} = \frac{1}{2}\},$$

where U_k is the characteristic function of the interpretation of U in $\mathcal{A}_k$.

Thus we see that the definition of $\leq_{a.s.}$ that we have adopted is the strongest among the possible variants of $\leq_{a.e.}$ in our framework.

Similar to the features of $\leq_{w.a.e.}$ we get:

Proposition 5.5 *If $\mathcal{L} \leq_{w.a.s.} \mathcal{L}'(\mu)$ and $\mathcal{L}'$ has the strong 0-1 law for μ, then so also does $\mathcal{L}$.*

Proof: For every sentence $\phi \in \mathcal{L}$, there is a $\phi' \in \mathcal{L}'$, with $\mu(\underline{\phi \leftrightarrow \phi'}) = 1$.

If $\mu(\underline{\phi'}) = 1$, then

$$\mu(\underline{\phi}) \geq \mu(\underline{\phi' \wedge (\phi \leftrightarrow \phi')}) = \mu(\underline{\phi'} \cap \underline{\phi \leftrightarrow \phi'}) = 1.$$

Thus we get $\mu(\underline{\phi}) = 1$.

If $\mu(\underline{\neg\phi'}) = 1$, then we similarly get $\mu(\underline{\neg\phi}) = 1$. ∎

The positive fixpoint logic **IFP** and the partial fixpoint logic **PFP** (see [16]) are proper extension of the first order logic, but they are properly contained in the infinitary logic with finitely many variables $\mathcal{L}^{\omega}_{\infty\omega}$ as shown in [16]. In symbols we have:

$$\mathbf{FO} \subset \mathbf{IFP} \subseteq \mathbf{PFP} \subset \mathcal{L}^{\omega}_{\infty\omega}.$$

However, we have:

Theorem 5.6 *With respect to the uniform measure μ we have:*

$$\mathbf{FO} \equiv_{a.s.} \mathbf{IFP} \equiv_{a.s.} \mathbf{PFP} \equiv_{a.s.} \mathcal{L}^{\omega}_{\infty\omega}.$$

Proof: We only have to prove that $\mathcal{L}^{\omega}_{\infty\omega} \leq_{a.s.} \mathbf{FO}$. From Lemma 3.2 in [16] for each $k \in \mathbf{N}$ we get a first order sentence ψ_k, implied by the extension axioms (and thus has $\mu(\underline{\psi_k}) = 1$ as before), and witnessing the collapse of $\mathcal{L}^{k}_{\infty\omega}$ to $\mathcal{L}^{k}_{\omega\omega}$, i.e. for every $\phi(\mathbf{x}) \in \mathcal{L}^{k}_{\infty\omega}$, there is a $\phi'(\mathbf{x}) \in \mathcal{L}^{k}_{\omega\omega}$, with

$$\psi_k \subseteq \forall \mathbf{x}(\phi(\mathbf{x}) \leftrightarrow \phi'(\mathbf{x})).$$

As in the proof of Theorem 5.4 we can now use Lemma 5.3 again (applied to ψ_k) to get $\mathbf{C} \subseteq \mathbf{M}$ such that $\mu(\underline{\mathbf{C}}) = 1$ and for every $k \in \mathbf{N}$ the set difference $\mathbf{C} \setminus \psi_k$ is finite, and thus the set difference $\mathbf{C} \setminus \forall \mathbf{x}(\phi(\mathbf{x}) \leftrightarrow \phi'(\mathbf{x}))$ is also finite. Now we use the closure property of **FO** as before to get for every $\phi(\mathbf{x}) \in \mathcal{L}^{\omega}_{\infty\omega}$ a formula $\tilde{\phi}(\mathbf{x}) \in \mathbf{FO}$ such that

$$\mathbf{C} \subseteq \forall \mathbf{x}(\phi(\mathbf{x}) \leftrightarrow \tilde{\phi}(\mathbf{x})).$$

∎

From Proposition 5.5 we get:

Corollary 5.7 *The logics* **IFP**, **PFP** *and* $\mathcal{L}^{\omega}_{\infty\omega}$ *have the strong 0-1 law for the uniform measure.*

■

Note that this corollary can be directly proved from Corollary 3.9 in [16], which states that if $\phi \in \mathcal{L}^{k}_{\infty\omega}$, then either $\theta_k \models \phi$ or $\theta_k \models \neg\phi$, where θ_k is the conjunction of the finitely many extension axioms with atmost k variables, and thus $\mu(\underline{\theta_k}) = 1$.

If μ is not the uniform measure, the previous theorem may not hold. However, we can get a characterization that parallels Theorem 5.1 in [10] with a similar proof.

Theorem 5.8 *For an arbitrary product measure* μ *the following statements are equivalent:*

(1) **FO** $\equiv_{a.s.}$ **IFP** $\equiv_{a.s.}$ **PFP** $\equiv_{a.s.}$ $\mathcal{L}^{\omega}_{\infty\omega}$ (μ).

(2) For every k, there are finitely many minimal sentences $\psi_1, .., \psi_r \in \mathcal{L}^{k}_{\infty\omega}$*, such that* $\mu(\underline{\psi_1 \vee .. \vee \psi_r}) = 1$.

Proof: $(1 \Rightarrow 2)$ Assume that for every k, $\mathcal{L}^{k}_{\infty\omega} \leq_{a.s.}$ **FO**, witnessed by a class **C** of strong measure 1. Assume that there are infinitely many minimal sentences in $\mathcal{L}^{k}_{\infty\omega}$ intersecting **C**. Then, taking countable unions of them we can get uncountably many sentences in $\mathcal{L}^{k}_{\infty\omega}$ that disagree on **C**, a contradiction since **FO** is countable. Thus, there are finitely many minimal sentences $\psi_1, .., \psi_r$ in $\mathcal{L}^{k}_{\infty\omega}$, such that **C** is a subset of the union $\psi_1 \vee .. \vee \psi_r$, which then must have strong measure 1.

$(2 \Rightarrow 1)$ Assuming (2), for every k we let $\mathbf{C} = \psi_1 \vee .. \vee \psi_r$. Now we know from [1] that every minimal sentence $\psi_i \in \mathcal{L}^{k}_{\infty\omega}$ is equivalent to a first order sentence. Thus on **C** **FO** will have the same expressive power as $\mathcal{L}^{k}_{\infty\omega}$. ■

The last theorem can be phrased in an abstract manner, which then looks like a compactness statement.

Corollary 5.9 *(of the proof) Let $\mathcal{L}$ be a countable logic that can express its equivalence classes, i.e. its minimal sentences union up to* $\mathbf{M}$. *Also let $\sigma(\mathcal{L})$ be the σ-algebra generated by $\mathcal{L}$ (here we only have to take countable unions of minimal sentences). Then for an arbitrary product measure μ the following statements are equivalent:*

(1) $\mathcal{L} \equiv_{a.s.} \sigma(\mathcal{L})$.

(2) There are finitely many minimal sentences $\psi_1, .., \psi_r \in \mathcal{L}$ (or $\sigma(\mathcal{L})$), such that $\mu(\underline{\psi_1 \vee .. \vee \psi_r}) = 1$.

∎

We also can get easy results on *a.s.* equivalences of logics that strengthen their corresponding a.e. results. For example, the next theorem is a stronger version of Theorem 4.5 in [10].

Theorem 5.10 *If the underlying vocabulary ν is nonunary, then with respect to the uniform measure:*

$$\mathcal{L}^{\omega}_{\infty\omega}(\mathbf{D}_2) \equiv_{a.s.} \mathcal{L}_{\infty\omega},$$

where $\mathbf{D}_2$ is the even quantifier which can express that a size of a definable subset of the model is even.

Proof: We only have to check that the class $\mathbf{C}$, on which the two logics collapse has $\mu(\underline{\mathbf{C}}) = 1$, which reduces to checking that $\sum_n (1 - \mu_n(\mathbf{C})) < \infty$. But in the proof of the original theorem this class has a probability of rejection $\mathcal{O}(n^{3/2}2^{-n/2})$. Therefore, the sequence $(1 - \mu_n(\mathbf{C}))$ is summable. ∎

Also for the almost sure descriptive complexity we get a strengthening of Corollary 4.8 in [10].

Corollary 5.11 *If the underlying vocabulary ν is nonunary, then with respect to the uniform measure:*

$$\mathbf{IFP}(\mathbf{D}_2) \equiv_{a.s.} \mathbf{PTIME},$$

and

$$\mathbf{PFP}(\mathbf{D}_2) \equiv_{a.s.} \mathbf{PSPACE}.$$

∎

Chapter 6

Other strong 0-1 laws.

6.1 Easy strong 0-1 laws.

As we saw before one can search in the literature for existing proofs for 0-1 laws and check if the proofs actually lead to the strong 0-1 law. The key tool here is Proposition 2.3 where a class $\mathbf{C}$ has strong measure 0 iff the sequence $\mu_n(\mathbf{C})$ is summable. Here is a list of some easy results about strong 0-1 laws with essentially the same proofs of the regular 0-1 laws.

Proposition 6.1 *The following logics have the strong 0-1 laws for the uniform measure on the set* $\mathbf{M}$ *of all finite models:*

(1) : $\Sigma^1_1(\exists^*\forall^*)$ *The existential second order logic with its first order part in the Bernays-Schönfinkel prefix class* $\exists^*\forall^*$ *(i.e., the existential quantifiers precede the universal quantifiers). See [14].*

(2) : $\Sigma^1_1(\exists^*\forall\exists^*)$ *The existential second order logic with its first order part in the Ackermann prefix class* $\exists^*\forall\exists^*$ *(i.e., only one universal quantifier). See [15].*

(3) : *The noncritical Keisler's probability logic* $\mathcal{L}^-_{\omega P}$. *The logic* $\mathcal{L}_{\omega P}$, *introduced in [11], is the first order logic augmented with the probability quantifiers* $\exists^{\geq r}$ *for each rational* $r \in (0,1)$, *where the formula* $(\exists^{\geq r}\mathbf{y})\phi(\mathbf{x},\mathbf{y})$ *says that the fraction of*

the tuples $\mathbf{y}$ *in the model that satisfy the formula* $\phi(\mathbf{x}, \mathbf{y})$ *is* $\geq r$.

The noncritical segment $\mathcal{L}^{-}_{\omega P}$ *of* $\mathcal{L}_{\omega P}$ *has two restrictions. The first is that the probability quantification* $(\exists^{\geq r}\mathbf{y})\phi(\mathbf{x}, \mathbf{y})$ *is noncritical, i.e.* $r \neq \lim_n \mu_n(\phi(\mathbf{c}, \mathbf{d}))$ *(in the vocabulary augmented by the constants* $\mathbf{c}, \mathbf{d}$ *that made to replace the variables* $\mathbf{x}, \mathbf{y}$*).*

The second restriction is that whenever $(\exists^{\geq r}\mathbf{y})\phi(\mathbf{x}, \mathbf{y})$ *appears as subformula of a formula in* $\mathcal{L}^{-}_{\omega P}$*, each atomic subformula of* $\phi(\mathbf{x}, \mathbf{y})$ *must contain at least one of the variables in* $\mathbf{y}$.

[13] gives a proof for the simple fragment of $\mathcal{L}^{-}_{\omega P}$ *(where the probability quantifiers apply on single variables rather than tuples of variables). For the nonsimple case, see Corollary 8.14 below.*

Note that the first order logic with probability quantifiers does not have a 0-1 law if we allow critical subformulas. An easy example is the sentence $(\exists^{\geq 1/2}x)R(x)$ (in the simple unary vocabulary $\{R\}$), which has an asymptotic probability $\frac{1}{2}$.

If we further allow binary predicates we lose even the convergence property as seen in the following interesting example:

Example: Consider the sentence $\phi = (\exists x)((\exists^{\geq 1/2}y)R(x,y) \wedge (\exists^{\geq 1/2}y)\neg R(x,y))$. Since ϕ can be read as $(\exists x)(\exists^{=1/2}y)R(x,y)$, one can see that if n is odd then $\mu_n(\phi) = 0$. However, if n is even one can show that $\lim_n \mu_n(\phi) = 1$. (Hint: Using independence we get $1 - \mu_n(\phi) = (1 - \binom{n}{n/2}(1/2)^n)^n$. Then use Stirling's Formula to get an asymptotic upper bound of the right hand side of the form $\exp(-cn)$ for some positive constant c).

6.2 Compton's slow growing classes.

In [2], [3] and [5] Compton investigated the regular 0-1 laws for measures which are uniform on classes $\mathbf{C}$ that are closed under disjoint unions and component. He showed that if $\mathbf{C}$ is a nonfast smoothly growing class (i.e. $\lim_n \frac{nc_{n-1}}{c_n} > 0$, where $c_n = |\mathbf{C} \cap \mathbf{M}_n|$), then the 0-1 law for the first order logic (as well as the monadic second order logic) is equivalent to the fact that the class $\mathbf{C}$ is slow growing (i.e. $\lim_n \frac{nc_{n-1}}{c_n} = \infty$).

The presence of n in the fraction $\frac{nc_{n-1}}{c_n}$ comes from considering the exponential generating series $c(x)$ of the class $\mathbf{C}$ defined by

$$c(x) = \sum_n \frac{c_n}{n!} x^n.$$

The proof of Compton's theorem can be strengthened to show that the slow growing condition actually implies the strong 0-1 law. The next theorem strengthens Theorem 6.3 in [5].

Theorem 6.2 *Let* $\mathbf{C} \subseteq \mathbf{M}$ *be a nonfast smoothly growing class of models that is closed under disjoint unions and components, and let* μ *be the uniform measure on* $\mathbf{C}$*. The following statements are equivalent:*

(1): *The class* $\mathbf{C}$ *is slow growing.*

(2): *The first order logic has the 0-1 law for* μ*.*

(3): *The monadic second order logic has the 0-1 law for* μ*.*

(2'): *The first order logic has the strong 0-1 law for* μ*.*

(3'): *The monadic second order logic has the strong 0-1 law for* μ*.*

Proof: Using Theorem 6.3 in [5], we only have to show that $(\mathbf{1}) \Rightarrow (\mathbf{3'})$.

Since $\lim_n \frac{nc_{n-1}}{c_n} = \infty$, we can get a nondecreasing function $f(n)$ such that

$$\frac{nc_{n-1}}{c_n} \geq f(n) \text{ , and } \lim_n f(n) = \infty.$$

Now the key of the proof lies in the next lemma which appears as Lemma 2.1(iii) in [3].

Lemma 6.3 *Let $\mathcal{B}$ be a connected model of size m with number of symmetries $\sigma(\mathcal{B})$ and let $\theta_{\mathcal{B},j}$ be the first order sentence which says that there are exactly j components of the model isomorphic to $\mathcal{B}$. Also let $b_{j,n}$ denote the number of models of size n, for which the sentence $\theta_{\mathcal{B},j}$ holds. Then we have that:*

$$b_{j,n} = (j+1)\sigma(\mathcal{B})\frac{n!}{(n+m)!}b_{j+1,n+m}.$$

∎

In Theorem 5.6 in [3], Compton applied this lemma once to prove that $\lim_n \mu_n(\theta_{\mathcal{B},j}) = 0$. Here we have to do better than that since our aim is to prove that $\sum_n \mu_n(\theta_{\mathcal{B},j}) < \infty$. For this we apply the lemma n times and get:

$$b_{j,n} = ((j+1)\sigma(\mathcal{B}))^n \, \frac{n!}{(n+nm)!} b_{j+n,n+nm}$$

Now we get:

$$\begin{aligned} \mu_n(\theta_{\mathcal{B},j}) = \frac{b_{j,n}}{c_n} &= ((j+1)\sigma(\mathcal{B}))^n \, \frac{n! b_{j+n,n+nm}}{(n+nm)! c_n} \\ &\leq ((j+1)\sigma(\mathcal{B}))^n \, \frac{n! c_{n+nm}}{(n+nm)! c_n} \\ &= ((j+1)\sigma(\mathcal{B}))^n \prod_{i=1}^{nm} \frac{c_{n+i}}{(n+i)c_{n+i-1}} \end{aligned}$$

$$
\begin{aligned}
&\leq ((j+1)\sigma(\mathcal{B}))^n \prod_{i=1}^{nm} \frac{1}{f(n+i)} \\
&\leq \left(\frac{(j+1)\sigma(\mathcal{B})}{(f(n))^m}\right)^n, \\
&\leq \left(\frac{1}{2}\right)^n,
\end{aligned}
$$

for sufficiently large n.

Thus we get that $\sum_n \mu_n(\theta_{\mathcal{B},j}) < \infty$. So, using Proposition 2.3 we get: $\mu(\overline{\theta_{\mathcal{B},j}}) = 0$.

Thus it follows that for a fixed connected model $\mathcal{B}$ and a fixed natural number l, $\mu(\overline{\bigvee_{j=0}^{l} \theta_{\mathcal{B},j}}) = 0$. In other words if we randomly choose a sequence of models $\langle \mathcal{A}_n \rangle$, where $\mathcal{A}_n$ has size n, we'll almost surely eventually end up with models that have more than l components isomorphic to $\mathcal{B}$.

Now fix a sentence ϕ of quantifier depth t and consider the relation $\mathcal{A} \approx_t \mathcal{A}'$, denoting that the models $\mathcal{A}$ and $\mathcal{A}'$ agree on all monadic second order sentences of quantifier depth $\leq t$. Clearly for each t, $\approx_t$ is an equivalence relation with finitely many equivalence classes. Among the class of connected models in **C** we choose the models $\mathcal{B}_1, \mathcal{B}_2, .., \mathcal{B}_k$ to represent the equivalence classes.

Using Ehrenfeucht-Fraïssé games for monadic second order logic, one can show that (see [2]) there is a number l such that if for each $i = 1, .., k$, the models $\mathcal{A}, \mathcal{A}'$ contain more than l components isomorphic to $\mathcal{B}_i$, then $\mathcal{A} \approx_t \mathcal{A}'$.

From the remark above we know that the models $\mathcal{A}_n$ will eventually a.s. contain more than l components isomorphic to $\mathcal{B}_i$ for each i. Thus either $\mathcal{A}_n \models \phi$ eventually a.s. or $\mathcal{A}_n \models \neg\phi$ eventually a.s., establishing the strong 0-1 law for μ. ∎

Examples: Using techniques of [9] one can show that the following classes of models are slow growing (see [3]). Thus, they have the strong 0-1 law for the monadic second order logic.

(1) Any class generated by finitely many connected components.

(2) The class of oriented forests of height 1.

(3) The class of equivalence relations.

(4) The class of partitions with selected subsets.

6.3 Sparse unary predicates.

Let's assume that the vocabulary ν contains one ternary predicate C and one unary predicate U. We restrict our class of models to be those for which the predicate C is interpreted as the cyclic relation over the universe $\{1,..,n\}$, i.e. $C(i,j,k)$ holds iff $i<j<k$ or $j<k<i$ or $k<i<j$.

Also we use the measure μ_n induced by independent Bernoulli trials with probability $p(n)$ to determine for each $i \in \{1,..,n\}$ whether $U(i)$ holds or not.

Thus we view our models as cycles of length n where $C(i,j,k)$ says that starting from i and going clockwise we meet j before k, and U is a random subset of $\{1,..,n\}$ with $i \in U$ holding independently with probability $p(n)$.

In [22] Shelah and Spencer investigated the presence of the 0-1 law in this class with this measure and showed that the 0-1 law holds iff both the probability function $p(n)$ and its complement $q(n) = 1 - p(n)$ fall between the "cracks" $n^{-1}, n^{-1/2}, n^{-1/3}, \ldots$ In particular if $p(n) = n^{-\alpha}$ where $\alpha > 0$, then the 0-1 law holds iff $\alpha \notin \{\frac{1}{k} : k \geq 1\}$.

The situation for the strong 0-1 law is much simpler. Here there are no "cracks" anymore as seen in the following theorem.

Theorem 6.4 *In the above class of random unary predicates with the ternary cyclic relation a strong 0-1 law holds iff either*

$(\mathbf{0}) : \sum_n np(n) < \infty$, *or*

$(\overline{\mathbf{0}}) : \sum_n nq(n) < \infty$, *or*

$(\mathbf{1})$: *For every* $\epsilon > 0$, *both* $p(n), q(n) > 1/n^\epsilon$ *for sufficiently large* n.

Note that the statements $(\mathbf{0})$, $(\overline{\mathbf{0}})$ and $(\mathbf{1})$ are mutually disjoint, since e.g. $(\mathbf{0})$ implies that $p(n) < 1/n$ for sufficiently large n.

Thus as a special case we get:

Corollary 6.5 *If the measure* μ_n *is induced by independent probability* $p(n) = n^{-\alpha}$ *with* $\alpha > 0$, *then a strong 0-1 law holds iff* $\alpha > 2$.

∎

To prove the theorem we need the following definition and some lemmas.

Definition 6.6 *A word* w *over the alphabet* $\{0,1\}$ *is a finite sequence of 0s and 1s, i.e.* $w : \{1,..,m\} \to \{0,1\}$. $|w| = m$ *is called the length of* w. *Restricting ourselves to the models of the above class we say that* w *is embedded in the model* A *starting from* i *(written* $w \sqsubseteq_i A$*) if starting from* i *(where* $1 \leq i \leq n = |A|$*) and going clockwise the word* w *codes the unary relation* U *on the elements* $i \oplus 1, .., i \oplus |w|$, *where* $\oplus$ *is the cyclic sum on* $\{1,..,n\}$. *Precisely speaking for* $1 \leq j \leq |w|$, $w(j) =$

$U(i \oplus j)$. We say that w is embedded in A (written $w \sqsubseteq A$) if for some $i \in A$, $w \sqsubseteq_i A$.

For a word w, we let ϕ_w be the first order sentence expressing $(w \sqsubseteq A)$.

Lemma 6.7 *Let μ_n be the measure on the set of cycles $\langle \{1, .., n\}, C \rangle$ with the unary predicate U induced by independent probability $p(n)$ and let $q(n)$ denote $1 - p(n)$. Also let w be a fixed word of length m with k 0s and l 1s (thus $m = k + l$). Then we get*

$(\mathbf{1}) : \mu_n(\phi_w) \leq np^k(n)q^l(n).$

Also if $np^k(n)q^l(n)/(2m) \leq 1$, we get

$(\mathbf{2}) : \mu_n(\phi_w) \geq np^k(n)q^l(n)/(4m).$

Proof: For $(\mathbf{1})$ writing the (event) ϕ_w as a union of overlapping events we get:

$$\phi_w = \bigcup_{i \in A} (w \sqsubseteq_i A).$$

Thus as an upper bound we have:

$$\mu_n(\phi_w) \leq \sum_{i \in A} \mu_n(w \sqsubseteq_i A) = np^k q^l.$$

Also for $(\mathbf{2})$ taking a smaller union of independent events we get

$$\phi_w \supseteq \bigcup_{q=0}^{[n/m]-1} (w \sqsubseteq_{qm+1} A).$$

Thus as a lower bound we have:

$$\begin{aligned} \mu_n(\phi_w) &\geq 1 - \prod_{q=0}^{[n/m]-1} (1 - \mu_n(w \sqsubseteq_{qm+1} A)) \\ &= 1 - \left(1 - p^k q^l\right)^{[n/m]} \end{aligned}$$

$$\begin{aligned} &\geq 1 - \exp(-[n/m]p^k q^l) \\ &\geq 1 - \exp(-np^k q^l/(2m)) \\ &\geq np^k q^l/(4m), \end{aligned}$$

where the last inequality is guaranteed by the smallness of the exponent. ■

The following lemma is an immediate consequence of Theorem 2.10 in [22].

Lemma 6.8 *For every $t \in \mathbf{N}$, there is a word w (called a persistent word for t) such that for every sentence ψ of quantifier depth $\leq t$, either $\phi_w \models \psi$ or $\phi_w \models \neg\psi$.*

■

Proof of the theorem:

We first prove that

(**1**) : For every $\epsilon > 0$, both $p(n), q(n) > 1/n^\epsilon$ for sufficiently large n,

implies the strong 0-1 law. For each first order sentence ψ of quantifier depth t we consider the sentence ϕ_w where w is the persistent word given by Lemma 6.8. By this lemma it suffices to prove that ϕ_w has strong measure 1, since then either ψ has strong measure 1 or $\neg\psi$ has strong measure 1, in which case ψ will have strong measure 0. We let k, l denote the number of 1s and 0s respectively in the word w.

As in the proof of Lemma 6.7 we have the estimate:

$$\mu_n(\neg\phi_w) \leq \exp(-np^k q^l/(2m)) \leq \exp(-n^{1-\epsilon m}/(2m)).$$

Choosing ϵ small enough, we can force the right hand side to be summable. Thus $\neg\phi_w$ has strong measure 0 and ϕ_w has strong measure 1 as claimed.

In the rest of the proof after considering each statement (**i**) ($1 \leq i < 4$), we'll assume the statement $\neg$(**i**) together with a new statement (**i + 1**). When $i = 4$ we'll assume $\neg$(**4**), and then consider the cases $\neg$(**0**) and (**0**).

So we'll start assuming

$\neg$(**1**) : There is an $\epsilon > 0$ and an infinite $J \subseteq \mathbf{N}$ such that either $p(n) \leq 1/n^{\epsilon}$ for every $n \in J$ or $q(n) \leq 1/n^{\epsilon}$ for every $n \in J$.

Without loss of generality we'll only assume the former case. In the latter case we just have to deal with $q(n)$ instead of $p(n)$ and to replace all the cycles considered with their complements. This will end the proof with Case ($\overline{\mathbf{0}}$) instead of (**0**).

Now consider the word w_k consisting of just k 1s. Taking k big enough to guarantee that $\epsilon k - 1 > 0$, we apply Lemma 6.7 to get that:

$$\mu_n(\phi_{w_k}) \leq np^k \leq \frac{1}{n^{\epsilon k - 1}}.$$

So that $\lim_{n \in J} \mu_n(\phi_{w_k}) = 0$.

If, however, we have

(**2**) : For every $\epsilon > 0$ we have an infinite $I \subset \mathbf{N}$ such that $p(n) > 1/n^{\epsilon}$ for every $n \in I$,

then, applying the same argument for (**1**), we get for every persistent word w an infinite $I \subset \mathbf{N}$ such that $\lim_{n \in I} \mu_n(\phi_w) = 1$.

Using Theorem 2.9 in [22] we know that for some persistent w the sentence ϕ_{w_k} is a consequence of ϕ_w.

But this contradicts even the regular 0-1 law for the sentence ϕ_{w_k}. Thus we assume

$\neg(\mathbf{2})$: There is an $\epsilon > 0$ such that $p(n) \leq 1/n^\epsilon$ for sufficiently large n.

We'll further assume

$(\mathbf{3})$: There is a $c > 0$ such that $p(n) \geq c/(n^2 \ln(n+2))$, for sufficiently large n.

Let k be the smallest natural number such that $1/k < \epsilon$ and for $i = 1, .., k$ let

$$J_i = \{n \in \mathbf{N} : \frac{c}{(n^2 \ln(n+2))^{1/i}} \leq p(n) \leq \frac{c}{(n^2 \ln(n+2))^{1/(2i)}}\}$$

Since for $i \geq 1$, $1/(2i) < 1/(i+1)$ we can check that for sufficiently large n we have that $n \in \bigcup_{i=1}^k J_i$.

From calculus we know that $\sum_n \frac{1}{n \ln(n+2)} = \infty$, and as the J_is exhaust all but finitely many natural numbers, for some $i \leq k$ we also have

$$\sum_{n \in J_i} \frac{1}{n \ln(n+2)} = \infty.$$

For such an i we apply Lemma 6.7 on the sentence ϕ_{w_i} to get that for sufficiently large $n \in J_i$,

$$\mu_n(\phi_{w_i}) \leq np^i \leq n \left(\frac{c}{(n^2 \ln(n+2))^{1/(2i)}} \right)^i = \frac{c^i}{(\ln(n+2))^{1/2}}.$$

Thus $\lim_{n \in J_i} \mu_n(\phi_{w_i}) = 0$, so $\mu(\underline{\phi_{w_i}}) = 0$.

Also for sufficiently large $n \in J_i$,

$$\mu_n(\phi_{w_i}) \geq \frac{np^i}{4m} \geq \frac{n}{4m} \left(\frac{c}{(n^2 \ln(n+2))^{1/i}} \right)^i = \frac{c^i}{4mn \ln(n+2)}.$$

Thus $\sum_{n \in J_i} \mu_n(\phi_{w_i}) = \infty$, so $\mu(\overline{\phi_{w_i}}) = 1$ violating the strong 0-1 law.

Hence we assume

$\neg(\mathbf{3})$: For every $c > 0$ we have $p(n) < c/(n^2 \ln(n+2))$ for infinitely many n,

and we further assume

$(\mathbf{4})$: There is a $d > 0$ such that $p(n) \geq d/n$ for infinitely many n.

Using $\neg(\mathbf{3})$, we apply Lemma 6.7 on the sentence ϕ_{w_1}, which says that U is not empty. We then see that $\mu_n(\phi_{w_1}) \leq c/(n \ln(n+2))$ for infinitely many n, hence $\mu_n(\phi_{w_1})$ converges to 0 on a subsequence.

On the other hand, using $(\mathbf{4})$, we get a lower bound for $\mu_n(\phi_{w_1})$ on another subsequence as follows:

$$\mu_n(\phi_{w_1}) = 1 - (1-p)^n \geq 1 - \exp(-pn) \geq 1 - \exp(-d),$$

again violating the regular 0-1 law.

At last we assume

$\neg(\mathbf{4})$: For every $d > 0$ we have $p(n) < d/n$ for sufficiently large n, i.e. $p(n) \in o(1/n)$.

We can then apply Lemma 6.7 once more on the sentence ϕ_{w_1} to get that for sufficiently large n,

$$\frac{1}{4} np \leq \mu_n(\phi_{w_1}) \leq np.$$

First we further assume

$\neg(\mathbf{0})$: $\sum_n np = \infty$.

Then $\sum_n \mu_n(\phi_{w_1}) = \infty$, so $\mu(\overline{\phi_{w_1}}) = 1$. But we also have $\lim_n \mu_n(\phi_{w_1}) = 0$, so $\mu(\underline{\phi_{w_1}}) = 0$ violating the strong 0-1 law.

Now we assume

$(\mathbf{0})$: $\sum_n np < \infty$.

Then $\sum_n \mu_n(\phi_{w_1}) < \infty$. Thus $\mu(\overline{\phi_{w_1}}) = 0$ and consequently $\mu(\underline{\neg\phi_{w_1}}) = 1$.

In other words, for a sample point $\langle \mathcal{A}_n \rangle$ in Ω, $\mathcal{A}_n \models \neg\phi_{w_1}$ eventually almost surely.

But using Ehrenfeucht-Fraïssé games one can show that, given a natural number t, for any two models $\mathcal{A}, \mathcal{B}$ of size $\geq 2^{t-1}$, if $\mathcal{A}, \mathcal{B}$ are models of the sentence $\neg\phi_{w_1}$ (so that their interpretation of U is empty) then $\mathcal{A} \equiv_t \mathcal{B}$, i.e. they agree on all sentences of quantifier depth $\leq t$. (Duplicator wins the t-move game by preserving the cyclic order of the elements chosen and distances $< 2^{t-i}$ at move i).

Thus, for any sentence ϕ, either $\mathcal{A}_n \models \phi$ eventually almost surely, or $\mathcal{A}_n \models \neg\phi$ eventually almost surely. So the strong 0-1 law holds in this case, finishing the proof of the theorem. ∎

6.4 Sparse random graphs.

In this section the results and the proofs are very similar to those of the previous section, though a little bit more involved. It's an interesting problem to find a unified way of treating both subjects.

Here we assume that the vocabulary ν contains only one binary predicate symbol R. Our restricted class will be those models satisfying the sentences $\forall x \forall y (Rxy \rightarrow Ryx)$ and $\forall x \neg Rxx$.

Thus a model can be viewed as a loopfree undirected graph with Rxy meaning that there is an edge between x and y.

In [21] Shelah and Spencer investigated the presence of the 0-1 law in the class of graphs if the measure μ_n is induced by independent Bernoulli trials with probability $p(n)$ for each edge. They showed that if $p(n) = n^{-\alpha}$ where $\alpha > 0$, then the 0-1 law holds iff

$$\alpha \notin (Q \cap (0,1]) \cup \{1 + \frac{1}{k} : k \geq 1\}.$$

Luczak and Spencer [17] further gave a nearly complete characterization of those $p(n)$ for which the 0-1 law holds.

The situation for the strong 0-1 law is radically different and very much simplified. Here we can actually give a complete characterization of when the strong 0-1 law holds.

Theorem 6.9 *Let the measure μ_n be induced by independent edge probability $p(n)$, and let $q(n)$ denote $(1 - p(n))$. Then a strong 0-1 law holds iff either*

$(\mathbf{0})$: $\sum_n n^2 p(n) < \infty$, *or*

$(\overline{\mathbf{0}})$: $\sum_n n^2 q(n) < \infty$, *or*

$(\mathbf{1})$: *For every $\epsilon > 0$, both $p(n), q(n) > 1/n^{\epsilon}$ for sufficiently large n.*

Note that the statements $(\mathbf{0})$, $(\overline{\mathbf{0}})$ and $(\mathbf{1})$ are mutually disjoint, since e.g. $(\mathbf{0})$ implies that $p(n) < 1/n^2$ for sufficiently large n.

Now we immediately get:

Corollary 6.10 *If the measure μ_n is induced by independent edge probability $p(n) = n^{-\alpha}$ with $\alpha > 0$, then a strong 0-1 law holds iff $\alpha > 3$.*

∎

To prove the theorem we need some definitions and a technical lemma.

Definition 6.11 *A graph G is a finite set of vertices $V(G)$ together with a set of edges $E(G)$ connecting the vertices (an edge can be viewed as a doubleton $\{x, y\}$, where $x, y \in V(G)$ are distinct vertices).*

Two graphs G, G' are isomorphic (written $G \simeq G'$) if there is a bijection between $V(G)$ and $V(G')$ that preserves the edges.

A graph H is a subgraph of G (written $H \subseteq G$) if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ as sets (note that some potential edges can be missing, so this definition deviates from the definition of a submodel in logic).

H is isomorphically embedded in G (written $H \sqsubseteq G$) if $H \simeq H'$ for some $H' \subseteq G$.

If $X \subseteq V(G)$ is a set of vertices in G, then $G[X]$ is the largest subgraph H of G with $V(H) = X$, i.e. $E(H) = E(G) \cap [X]^2$ ($G[X]$ is then a submodel of G). G is said to be balanced if

$$\frac{|E(G)|}{|V(G)|} \geq \frac{|E(H)|}{|V(H)|}$$

for each $H \subseteq G$.

Lemma 6.12 *Let μ_n be the measure on the set of graphs with universe $\{1, .., n\}$ induced by independent edge probability $p(n)$. Let H be a fixed balanced graph with v vertices and e edges, and let $\phi_H = (H \sqsubseteq G)$ be the first order sentence stating that H is isomorphically embedded in the finite graph G. Let $J \subseteq \mathbf{N}$ be some infinite set of natural numbers and assume that $\lim_{n \in J} n^v p^e(n) = 0$. Then for sufficiently large $n \in J$ we get*

$$\frac{n^v p^e(n)}{2v!} \leq \mu_n(\phi_H) \leq n^v p^e(n).$$

Proof: Writing the (event) ϕ_H as a union of overlapping events we get:

$$\phi_H = \bigcup_{\substack{X \subset V(G),\\ |X|=v}} (H \sqsubseteq G[X]).$$

Thus as an upper bound we have:

$$\mu_n(\phi_H) \leq \sum_{\substack{X \subset V(G),\\ |X|=v}} \mu_n(H \sqsubseteq G[X]) \leq \binom{n}{v} v! p^e \leq n^v p^e.$$

Also as a lower bound we have:

$$\begin{aligned} \mu_n(\phi_H) &\geq \sum_{\substack{X \subset V(G),\\ |X|=v}} \mu_n(H \sqsubseteq G[X]) - \sum_{\substack{X,Y \subset V(G),\\ |X|=|Y|=v, X \neq Y}} \mu_n(H \sqsubseteq G[X], G[Y]) \\ &\geq \binom{n}{v} p^e - \sum_{k<v} \left(\sum_{\substack{X,Y \subset V(G),\\ |X|=|Y|=v, |X \cap Y|=k}} \mu_n(H \sqsubseteq G[X], G[Y]) \right) \\ &\geq \frac{(n)_v}{v!} p^e - \sum_{k<v} n^{2v-k} (v!)^2 p^{2e-e'}, \end{aligned}$$

where $e' = e'(k)$ is the maximum number of edges in subgraphs of H with k vertices. But using the fact that H is balanced (i.e. $\frac{e'}{k} \leq \frac{e}{v}$) we get

$$\begin{aligned} n^{2v-k} p^{2e-e'} &= (n^v p^e)^2 \left(n p^{e'/k}\right)^{-k} \\ &\leq (n^v p^e)^2 \left(n p^{e/v}\right)^{-k} = (n^v p^e)^{2-k/v} \leq (n^v p^e)^{1+1/v}. \end{aligned}$$

Thus

$$\mu_n(\phi_H) \geq \frac{n^v p^e}{v!} \left(\frac{(n)_v}{n^v} - (n^v p^e)^{1/v} (v!)^3 v \right) \geq \frac{n^v p^e}{2v!},$$

for sufficiently large $n \in J$, completing the proof of the lemma. ∎

Proof of the theorem:

We first prove that

$(\mathbf{1})$: For every $\epsilon > 0$, both $p(n), q(n) > 1/n^\epsilon$ for sufficiently large n,

implies the strong 0-1 law. Here we just have to modify the proof of Theorem 3.4 which is based on Fagin's proof. There we replace each fraction 1/2 in the calculation of $\mu_n(\neg\psi)$ either by the edge probability $p(n)$ or by its complement $q(n)$. Then using $(\mathbf{1})$ we get the estimate:

$$\mu_n(\neg\psi) \leq n^m \left(1 - \left(\frac{1}{n^\epsilon}\right)^{2m+1}\right)^{n-m} \leq n^m \exp\left(-\frac{n-m}{n^{\epsilon(2m+1)}}\right).$$

Choosing ϵ small enough, we can force the right hand side to be summable. We then proceed as in Theorem 3.4 to prove the strong 0-1 law.

In the rest of the proof after considering each statement $(\mathbf{i})$ $(1 \leq i < 4)$, we'll assume the statement $\neg(\mathbf{i})$ together with a new statement $(\mathbf{i+1})$. When $i = 4$ we'll assume $\neg(\mathbf{4})$, and then consider the cases $\neg(\mathbf{0})$ and $(\mathbf{0})$.

So we'll start assuming

$\neg(\mathbf{1})$: There is an $\epsilon > 0$ and an infinite $J \subseteq \mathbf{N}$ such that either $p(n) \leq 1/n^\epsilon$ for every $n \in J$ or $q(n) \leq 1/n^\epsilon$ for every $n \in J$.

Without loss of generality we'll only assume the former case. In the latter case we just have to deal with $q(n)$ instead of $p(n)$ and to replace all the graphs considered with their complements. This will end the proof with the case $(\overline{\mathbf{0}})$ instead of $(\mathbf{0})$.

Now consider the (balanced) complete graph K_v with v vertices and $e = v(v-1)/2$ edges. Taking v big enough to guarantee that $\epsilon e - v > 0$, we see that

$$\lim_{n\in J} n^v p^e \leq \lim_{n\in J} n^v \left(\frac{1}{n^\epsilon}\right)^e = \lim_{n\in J} \frac{1}{n^{\epsilon e - v}} = 0.$$

We can now apply the lemma on K_v to get that $\lim_{n \in J} \mu_n(\phi_{K_v}) = 0$.

If, however, we have

$(\mathbf{2})$: For every $\epsilon > 0$ we have an infinite $I \subset \mathbf{N}$ such that $p(n) > 1/n^\epsilon$ for every $n \in I$,

then, applying the same argument for $(\mathbf{1})$, we get that $\lim_{n \in I} \mu_n(\psi) = 1$ for every extension axiom ψ that states the extendibility of each graph with new edges (and therefore has only $p(n)$ in the calculation of it's measure).

This contradicts even the regular 0-1 law, since the sentence ϕ_{K_v} is a consequence of such extension axioms. Thus we assume

$\neg(\mathbf{2})$: There is an $\epsilon > 0$ such that $p(n) \leq 1/n^\epsilon$ for sufficiently large n.

We'll further assume

$(\mathbf{3})$: There is a $c > 0$ such that $p(n) \geq c/(n^3 \ln(n+2))$, for sufficiently large n.

Now consider the following list of balanced graphs

$$(H_1, H_2, H_3, ..) = (K_2, K_3', L_4, K_3, K_4', K_4, K_5', K_5, ..),$$

where K_v is again the complete graph on v vertices, K_v' is the same as K_v with one edge removed, and L_4 is the linear graph on 4 vertices.

For each $i \in \mathbf{N}$ let $v_i = |V(H_i)|$ and $e_i = |E(H_i)|$ denote the number of vertices and edges respectively of the graph H_i, and let $a_i = v_i/e_i, b_i = (v_i + 1)/e_i$.

It's not hard to check that for every $i \in \mathbf{N}$ we have the inequalities:

$$a_{i+1} < a_i \leq b_{i+1} < b_i \ , \ e_{i+1} \leq 2e_i.$$

Also we have that $\lim_i a_i = 0$. So let k be the smallest natural number such that $a_k < \epsilon$.

Now for $i = 1, .., k$ let

$$J_i = \{n \in \mathbf{N} : \frac{c}{n^{b_i}(\ln(n+2))^{1/e_i}} \leq p(n) \leq \frac{c}{n^{a_i}(\ln(n+2))^{1/(2e_i)}}\}$$

Using our assumptions $\neg(\mathbf{2}), (\mathbf{3})$ and the inequalities above we can check that for sufficiently large n we have that $n \in \bigcup_{i=1}^{k} J_i$.

From calculus we know that $\sum_n \frac{1}{n \ln(n+2)} = \infty$, and as the J_is exhaust all but finitely many natural numbers, for some $i \leq k$ we also have

$$\sum_{n \in J_i} \frac{1}{n \ln(n+2)} = \infty.$$

For such an i we apply the lemma on the sentence ϕ_{H_i} to get that for sufficiently large $n \in J_i$,

$$\mu_n(\phi_{H_i}) \leq n^{v_i} p^{e_i} \leq n^{v_i} \left(\frac{c}{n^{a_i}(\ln(n+2))^{1/(2e_i)}} \right)^{e_i} = \frac{c^{e_i}}{(\ln(n+2))^{1/2}}.$$

Thus $\lim_{n \in J_i} \mu_n(\phi_{H_i}) = 0$, so $\mu(\underline{\phi_{H_i}}) = 0$.

Also for sufficiently large $n \in J_i$,

$$\mu_n(\phi_{H_i}) \geq \frac{n^{v_i} p^{e_i}}{2v_i!} \geq \frac{n^{v_i}}{2v_i!} \left(\frac{c}{n^{b_i}(\ln(n+2))^{1/e_i}} \right)^{e_i} = \frac{c^{e_i}}{2v_i! n \ln(n+2)}.$$

Thus $\sum_{n \in J_i} \mu_n(\phi_{H_i}) = \infty$, so $\mu(\overline{\phi_{H_i}}) = 1$ violating the strong 0-1 law.

Hence we assume

$\neg(\mathbf{3})$: For every $c > 0$ we have $p(n) < c/(n^3 \ln(n+2))$ for infinitely many n,

and we further assume

$(\mathbf{4})$: There is a $d > 0$ such that $p(n) \geq d/n^2$ for infinitely many n.

Using $\neg(\mathbf{3})$, we apply the lemma on the sentence ϕ_{K_2}, which states the existence of a complete subgraph on 2 vertices (i.e. the existence of an edge). We then see that $\mu_n(\phi_{K_2}) \leq c/(n \ln(n+2))$ for infinitely many n, hence $\mu_n(\phi_{K_2})$ converges to 0 on a subsequence.

On the other hand, using (**4**), we get a lower bound for $\mu_n(\phi_{K_2})$ on another subsequence as follows:

$$\mu_n(\phi_{K_2}) = 1 - (1-p)^{n(n-1)/2} \geq 1 - \exp(-pn^2/4) \geq 1 - \exp(-d/4),$$

again violating the regular 0-1 law.

At last we assume

$\neg$(**4**) : For every $d > 0$ we have $p(n) < d/n^2$ for sufficiently large n, i.e. $p(n) \in o(1/n^2)$.

We can then apply the lemma once more on the sentence ϕ_{K_2} to get that for sufficiently large n,

$$\frac{1}{4}n^2 p \leq \mu_n(\phi_{K_2}) \leq n^2 p.$$

First we further assume

$\neg$(**0**) : $\sum_n n^2 p = \infty$.

Then $\sum_n \mu_n(\phi_{K_2}) = \infty$, so $\mu(\overline{\phi_{K_2}}) = 1$. But we also have $\lim_n \mu_n(\phi_{K_2}) = 0$, so $\mu(\underline{\phi_{K_2}}) = 0$ violating the strong 0-1 law.

Now we assume

(**0**) : $\sum_n n^2 p < \infty$.

Then $\sum_n \mu_n(\phi_{K_2}) < \infty$. Thus $\mu(\overline{\phi_{K_2}}) = 0$ and consequently $\mu(\underline{\neg\phi_{K_2}}) = 1$.

But the sentence $\neg\phi_{K_2}$ (stating that the graph is empty) is $\omega-$categorical and hence complete. So a Fagin like argument establishes the strong 0-1 law in that case, finishing the proof of the theorem. ∎

Chapter 7

Analogues for Baire category.

After reading a draft of this thesis, K. Compton suggested that a topological analogue (in the Baire category sense) of the strong 0-1 law can be investigated.

The strong 0-1 law for a logic $\mathcal{L}$ states that for any sentence $\phi \in \mathcal{L}$ the two tail events $\underline{\phi}$ and $\overline{\phi}$ are either small together or big together in the measure sense, that is their measures coincide and are either 0 or 1. Of course the smallness or bigness of the tail events is guaranteed by the Kolmogorov's 0-1 Law.

In topological spaces that are big enough the small sets are the meager sets (the countable unions of nowhere dense sets), whereas the big sets are the comeager sets (the complements of meager sets). Some work toward Small-Big laws in the topological sense was done in [18] and [19].

In [20] J. Oxtoby proved an analogue of the Kolmogorov's 0-1 Law, which states that under certain conditions, any tail Borel set in a product Topological space is either meager or comeager, that is tail Borel sets are either small or big in the topological sense.

Let us equip each $\mathbf{M}_n$ with the natural discrete topology, and $\Omega = \prod_{n \in \mathbf{N}} \mathbf{M}_n$ with the product topology. Since each $\mathbf{M}_n$ has a finite discrete topology, the conditions mentioned above are satisfied. Also the tail sets

$$\underline{\phi} = \bigcup_N \bigcap_{n \geq N} \phi^n$$

and

$$\overline{\phi} = \bigcap_N \bigcup_{n \geq N} \phi^n$$

are F_σ and G_δ respectively. Thus Oxtoby's Small-Big Law says that each of them is either meager or comeager.

Thus some natural questions arise: Is there a strong small-big law for logics in the topological sense, i.e. is it true that for some logics the tail sets $\underline{\phi}$ and $\overline{\phi}$ are either meager together or comeager together? If this is true, do measure and topology agree, i.e. are the small sentences in the measure sense also small in the topological sense and vice versa?

The next proposition gives a somewhat disappointing answer to these questions, it says that if the logic is interesting enough, then we get a negative answer to the first question.

Proposition 7.1 *Let ϕ be a sentence such that both ϕ and $\neg\phi$ have infinite spectrums, and thus $\emptyset \neq \underline{\phi}, \overline{\phi} \neq \Omega$. Then $\underline{\phi}$ is meager and $\overline{\phi}$ is comeager.*

Proof: Since $\underline{\phi} = \Omega \setminus \overline{\neg\phi}$, it is enough to prove that $\overline{\phi}$ is comeager. But

$$\overline{\phi} = \bigcap_N \bigcup_{n \geq N} \phi^n.$$

Since the spectrum of ϕ is infinite, for each N we see that $\bigcup_{n \geq N} \phi^n$ is a dense open set. Thus $\overline{\phi}$ is a countable intersection of dense open sets, and is therefore comeager. ∎

Thus an analogue of the Strong 0-1 Law in topology can not be true if the logic has some interesting sentences. This adds more cases in which measure and topology do not agree.

Chapter 8

Strong convergence laws for formulas.

In Chapter 1 we defined for a sentence ϕ the "inner" and "outer" measures $\mu(\underline{\phi})$ and $\mu(\overline{\phi})$. We saw there that those measures take only the trivial values 0 or 1 as $\underline{\phi}$ and $\overline{\phi}$ are tail events.

In this chapter we'll define a generalization of those concepts for any abstract formula $\phi(\mathbf{x})$ so that $\mu(\underline{\phi(\mathbf{x})})$ and $\mu(\overline{\phi(\mathbf{x})})$ may take real values between 0 and 1.

Definition 8.1 *The probability of a formula $\phi(\mathbf{x})$ in a model $\mathcal{A}$, denoted by $P_{\mathcal{A}}(\phi(\mathbf{x}))$ is the fraction of tuples $\mathbf{x}$ in $\mathcal{A}$ that satisfy the formula $\phi(\mathbf{x})$, i.e. if $\mathcal{A}$ is of size n and $|\mathbf{x}| = k$ then*

$$P_{\mathcal{A}}(\phi(\mathbf{x})) = \frac{|\{\mathbf{a} \in \mathcal{A}^k : \mathcal{A} \models \phi[\mathbf{a}]\}|}{n^k}.$$

For simplicity, we just took a uniform distribution of the elements of the model $\mathcal{A}$.

Also we can use Keisler's probability quantifier $(\exists^{\geq r}\mathbf{x})$ mentioned in Section 6.1 to alternately write:

$$\begin{aligned} P_{\mathcal{A}}(\phi(\mathbf{x})) &= \sup\{r \in [0,1] : \mathcal{A} \models (\exists^{\geq r}\mathbf{x})\phi(\mathbf{x})\} \\ &= \inf\{r \in [0,1] : \mathcal{A} \not\models (\exists^{> r}\mathbf{x})\phi(\mathbf{x})\}. \end{aligned}$$

Note that if ϕ is a sentence, i.e. the sequence $\mathbf{x}$ is empty, then $P_{\mathcal{A}}(\phi) = 1$ if $\mathcal{A} \models \phi$ and 0 otherwise.

For a sequence $\mathbf{A} = \langle \mathcal{A}_n \rangle_{n \in \mathbf{N}}$ of models $\mathcal{A}_n$ of size n (which is an element of our sample space Ω), consider the tail random variables $\underline{\lim}_n P_{\mathcal{A}_n}(\phi(\mathbf{x}))$ and $\overline{\lim}_n P_{\mathcal{A}_n}(\phi(\mathbf{x}))$.

By Kolmogorov's 0-1 Law they must be constant almost surely. We then define the "inner measure" $\mu(\underline{\phi(\mathbf{x})})$ and the "outer measure" $\mu(\overline{\phi(\mathbf{x})})$ to be those constant values. Thus we can formally write:

$$\mu(\underline{\phi(\mathbf{x})}) = \mathbf{E}\left(\underline{\lim}_n \ P_{\mathcal{A}_n}(\phi(\mathbf{x}))\right)$$

and

$$\mu(\overline{\phi(\mathbf{x})}) = \mathbf{E}\left(\overline{\lim}_n \ P_{\mathcal{A}_n}(\phi(\mathbf{x}))\right),$$

where $\mathbf{E}$ denotes the expectation functional.

Note that, as in the case of inner and outer measures for sentences, we have

$$\mu(\overline{\phi(\mathbf{x})}) = 1 - \mu(\underline{\phi(\mathbf{x})}).$$

We actually didn't give any meaning for $\underline{\phi(\mathbf{x})}$ or $\overline{\phi(\mathbf{x})}$ and thus μ in $\mu(\underline{\phi(\mathbf{x})})$ and $\mu(\overline{\phi(\mathbf{x})})$ is to be viewed as a formal "measure". However, we'll allow ourselves to abuse notation by using the terms "inner" and "outer measures".

Using Keisler's probability quantifiers the next proposition gives an alternate definition of the inner and outer measure of formulas and relate them to those of sentences.

Proposition 8.2

$$\mu(\underline{\phi(\mathbf{x})}) = \sup\left\{r \in [0,1] : \mu\left(\underline{(\exists^{\geq r}\mathbf{x})\phi(\mathbf{x})}\right) = 1\right\},$$

and

$$\mu(\overline{\phi(\mathbf{x})}) = \inf\left\{r \in [0,1] : \mu\left(\overline{(\exists^{>r}\mathbf{x})\phi(\mathbf{x})}\right) = 0\right\}.$$

■

Using this proposition, we can easily check that the definition of the inner and outer measures for a formula $\phi(\mathbf{x})$ reduces to the old definition if ϕ is a sentence.

In the literature the underlying relational vocabulary ν is sometimes extended by a list of distinct constant symbols $\mathbf{c}$ (with $|\mathbf{c}| = |\mathbf{x}|$) to get the vocabulary $\nu \cup \mathbf{c}$. Thus a $(\nu \cup \mathbf{c})$-model will have an interpretation of the constants $\mathbf{c}$ in the model.

If we then replace $\mathbf{x}$ by $\mathbf{c}$ in $\phi(\mathbf{x})$ we can talk about the measure $\mu_n(\phi(\mathbf{c}))$ of the sentence $\phi(\mathbf{c})$, where for simplicity we take the distribution of the interpretation of the tuple $\mathbf{c}$ in a given model to be uniform.

Fixing a product measure μ, we say that a logic $\mathcal{L}$ has the convergence law for formulas if for every formula $\phi(\mathbf{x}) \in \mathcal{L}$, $\lim_n \mu_n(\phi(\mathbf{c}))$ exists.

It follows from Theorem 7 in [7] that, taking μ to be the uniform product measure, the first order logic has the convergence law for formulas. In fact, for any first order formula $\phi(\mathbf{x})$, $\lim \mu_n(\phi(\mathbf{c}))$ exists and equals to $l/2^s$, where l and s are positive integers determined by the form of $\phi(\mathbf{x})$.

Actually in [7] the constants $\mathbf{c}$ in $\phi(\mathbf{c})$ are forced to be interpreted by distinct elements in the model. However, in the limit this will give us the same result if we allow them to take common values.

The next proposition parallels Proposition 2.1 and relates the inner and outer measure of a formula $\phi(\mathbf{x})$ with the asymptotic probability of the sentence $\phi(\mathbf{c})$.

Proposition 8.3

$$\mu(\underline{\phi(\mathbf{x})}) \leq \underline{\lim_n}\ \mu_n(\phi(\mathbf{c})) \leq \overline{\lim_n}\ \mu_n(\phi(\mathbf{c})) \leq \mu(\overline{\phi(\mathbf{x})}).$$

■

In view of this proposition we have the following definition.

Definition 8.4 *For a fixed product measure μ a logic $\mathcal{L}$ has the strong convergence law for formulas if for every formula $\phi(\mathbf{x}) \in \mathcal{L}$,*

$$\mu(\underline{\phi(\mathbf{x})}) = \mu(\overline{\phi(\mathbf{x})}).$$

From Proposition 8.3 we easily get:

Corollary 8.5 *A logic that has the strong convergence law for formulas also has the convergence law for formulas.*

■

Also, considering only sentences in $\mathcal{L}$, the definition reduces to that of the strong 0-1 law, i.e. we have:

Corollary 8.6 *A logic that has the strong convergence law for formulas must also have the 0-1 law.*

■

Note that, unlike the case of the convergence law for sentences, which is weaker than the 0-1 law, the strong convergence law for formulas is actually stronger than the strong 0-1 law.

Let's try to have a close look at the strong convergence law. From the definition the collapse of $\mu(\underline{\phi(\mathbf{x})})$ and $\mu(\overline{\phi(\mathbf{x})})$ to a common value r means that the limit of

$P_{\mathcal{A}_n}(\phi(\mathbf{x}))$ almost surely exists and equals to r. In other words it says that for every small δ, $P_{\mathcal{A}_n}(\phi(\mathbf{x}))$ will almost surely eventually fall in the interval $[r-\delta, r+\delta]$.

In other words, using Keisler's probability quantifier, this is equivalent to saying that

$$\mu\left(\underline{(\exists^{\in[r-\delta,r+\delta]}\mathbf{x})\phi(\mathbf{x})}\right) = 1,$$

for every small δ.

Thus the strong convergence law implies that for sufficiently large n, in a strong way "most" of the models $\mathcal{A}_n$ of size n, will have the value $P_{\mathcal{A}_n}(\phi(\mathbf{x}))$ "close" to some constant value r.

The following proposition parallels Proposition 5.5 and relates the strong convergence law to the weakly almost surely equivalence of Logics.

Proposition 8.7 *If $\mathcal{L} \leq_{w.a.s.} \mathcal{L}'(\mu)$ and $\mathcal{L}'$ has the strong convergence law for formulas, then so also does $\mathcal{L}$.*

■

In this proposition the logic $\mathcal{L}'$ can be a fragment of the logic $\mathcal{L}$. If this fragment is the quantifier-free part or the Boolean combinations of some "basic" formulas, we say that $\mathcal{L}$ has the almost sure quantifier elimination.

Thus the proposition gives us a technique to prove that the strong convergence law holds for formulas in some logic $\mathcal{L}$. Namely, prove that some fragment $\mathcal{L}'$ thereof possesses the strong convergence law, then prove that $\mathcal{L}$ can be reduced to $\mathcal{L}'$ weakly almost surely.

In the following, Theorem 8.10, which is interesting in its own right, subsumes the strong 0-1 law and leads to the strong convergence law for first order formulas.

The proof is a rather elegant and much shorter version of Glebskii et. al.'s proof of the 0-1 law, see [7]. But first we need some definitions.

Definition 8.8 $\mathbf{FO}(\mathbf{x})$ *(*$\mathbf{FO_0}(\mathbf{x})$*) denotes the first order (quantifier-free) formulas with free variables in* $\mathbf{x}$*. Thus* $\mathbf{FO}()$ *denotes the first order sentences, and we let* $\mathbf{FO_0}()$ *(the quantifier-free sentences) include the two symbols* $\mathbf{T}$ *and* $\mathbf{F}$*, denoting the always true and always false sentences respectively.*

We say that the formula $\phi_1(\mathbf{x})$ *is equivalent to the formula* $\phi_2(\mathbf{x})$ *almost surely (with respect to* μ*) and write* $\phi_1(\mathbf{x}) \equiv \phi_2(\mathbf{x})$ *a.s.(*μ*) if*

$$\mu\left(\underline{\forall \mathbf{x}(\phi_1(\mathbf{x}) \leftrightarrow \phi_2(\mathbf{x}))}\right) = 1.$$

Thus $\phi_1(\mathbf{x}) \equiv \phi_2(\mathbf{x})$ *can be viewed as an abbreviation of* $\mathcal{A}_n \models \forall \mathbf{x}(\phi_1(\mathbf{x}) \leftrightarrow \phi_2(\mathbf{x}))$ *eventually.*

The following lemma is a slightly simpler version of Lemma 1 in [7].

Lemma 8.9 *Let* $\psi(\mathbf{x}, y)$ *be a first order quantifier-free formula where* y *appears in all of its atomic subformulas. Then, with respect to the uniform measure,*

$(\forall y)\psi(\mathbf{x}, y) \equiv \mathbf{T}$ *almost surely if* $\psi(\mathbf{x}, y) \equiv \mathbf{T}$ *(i.e.* $\psi(\mathbf{x}, y)$ *is identically true), and* $(\forall y)\psi(\mathbf{x}, y) \equiv \mathbf{F}$ *almost surely otherwise.*

■

The proof of the lemma needs the fact that y appears in all atomic subformulas, so that for fixed $\mathbf{x}$ and distinct y, y' different from $\mathbf{x}$, the formulas $\psi(\mathbf{x}, y)$ and $\psi(\mathbf{x}, y')$ are independent.

Note that the dual of this lemma (the one talking about existential quantification) is essentially the statement that Fagin's extension axioms have strong measure 1. Thus, although the two proofs of the 0-1 law given by Glebskii et. al. and Fagin are quite different, they are still close in spirit, as both of them are based on one fact.

Theorem 8.10 *With respect to the uniform measure,* $\mathbf{FO}(\mathbf{x}) \leq_{w.a.s.} \mathbf{FO_0}(\mathbf{x})$ *(first order logic has the almost sure quantifier elimination).*

That is, for every first order formula $\phi(\mathbf{x})$*, there is a first order quantifier-free formula* $\theta(\mathbf{x})$*, such that,*

$$\phi(\mathbf{x}) \equiv \theta(\mathbf{x}) \ a.s..$$

Proof: By induction on the complexity of $\phi(\mathbf{x})$.

Base(atomic formulas): Obvious.

Step: If $\phi(\mathbf{x})$ is a negation or a conjunction, the argument is easy. Thus we'll assume that $\phi(\mathbf{x})$ is of the form $(\forall y)\psi(\mathbf{x}, y)$. By the induction hypothesis we get a quantifier-free formula $\theta(\mathbf{x}, y)$ such that

$$\psi(\mathbf{x}, y) \equiv \theta(\mathbf{x}, y) \text{ a.s.}$$

Thus we get

$$\begin{aligned} \phi(\mathbf{x}) &= (\forall y)\psi(\mathbf{x}, y) \\ &\equiv (\forall y)\theta(\mathbf{x}, y) \text{ a.s.} \end{aligned}$$

Writing $\theta(\mathbf{x}, y)$ in the conjunctive normal form, distributing the quantifier on the conjunction, and taking out atomic formulas that do not contain the variable y, we get

$$\phi(\mathbf{x}) \equiv \bigwedge_{i=1}^{k} ((\forall y)\psi_i(\mathbf{x}, y) \vee \theta_i(\mathbf{x})) \text{ a.s.}$$

Now using Lemma 8.9, each $(\forall y)\psi_i(\mathbf{x}, y)$ is either equivalent to $\mathbf{T}$ a.s. or it's equivalent to $\mathbf{F}$ a.s.. Thus letting

$$I = \{i \in \{1, .., k\} : (\forall y)\psi_i(\mathbf{x}, y) \equiv \mathbf{F} \text{ a.s.}\},$$

we get that

$$\phi(\mathbf{x}) \equiv \bigwedge_{i \in I} \theta_i(\mathbf{x}) \text{ a.s..}$$

∎

Note that, taking $\mathbf{x}$ to be empty, the theorem says that each first order sentence collapses to $\mathbf{T}$ or $\mathbf{F}$ almost surely. This is the strong 0-1 law!

Now to prove the strong convergence law for first order formulas all we need is to prove it for quantifier free formulas. To do this we'll first strengthen Theorem 8.10 to include the noncritical Keisler's probability logic $\mathcal{L}^-_{\omega P}$ mentioned in Section 6.1.

The next lemma is similar to Lemma 8.9.

Lemma 8.11 *Let $\psi(\mathbf{x}, y)$ be a first order quantifier-free formula where y appears in all of its atomic subformulas. Then, with respect to the uniform measure,*

$(\exists^{\geq r} y)\psi(\mathbf{x}, y) \equiv \mathbf{T}$ *almost surely if* $r < \lim_n \mu_n(\psi(\mathbf{c}, d))$

and $(\exists^{\geq r} y)\psi(\mathbf{x}, y) \equiv \mathbf{F}$ *almost surely if* $r > \lim_n \mu_n(\psi(\mathbf{c}, d))$

∎

The proof of this lemma, which uses the central limit theorem, can be found in [13].

However, Lemma 8.11 is not enough to prove the almost sure quantifier elimination for the logic $\mathcal{L}^{-}_{\omega P}$, as we don't know how to deal with the quantification $(\exists^{\geq r}\mathbf{y})$, when $|\mathbf{y}| > 1$.

The next lemma strengthens Lemma 8.11, as it deals with this case.

Lemma 8.12 *Let $\psi(\mathbf{x},\mathbf{y})$ be a first order quantifier-free formula, in which each atomic subformula contains at least one of the variables in $\mathbf{y}$. Then, with respect to the uniform measure,*

$(\exists^{\geq r}\mathbf{y})\psi(\mathbf{x},\mathbf{y}) \equiv \mathbf{T}$ *almost surely if* $r < \lim_n \mu_n(\psi(\mathbf{c},\mathbf{d}))$

and $(\exists^{\geq r}\mathbf{y})\psi(\mathbf{x},\mathbf{y}) \equiv \mathbf{F}$ *almost surely if* $r > \lim_n \mu_n(\psi(\mathbf{c},\mathbf{d}))$

Proof: By induction on $|\mathbf{y}|$.

Base: $(|\mathbf{y}| = 1)$ This is Lemma 8.11.

Step: (say we quantify over $\mathbf{y}, z$)

Case 1: $r < r_0 = \lim_n \mu_n(\psi(\mathbf{c},\mathbf{d},e))$.

We assume that $\psi(\mathbf{x},\mathbf{y},z)$ is put in a full disjunctive normal form, so that the disjuncts are exclusive. We'll show by an example that if Case 1 is proved for the exclusive disjuncts, it will then follow that it holds for the full disjunction.

Say $\psi(\mathbf{x},\mathbf{y},z) = \psi_1(\mathbf{x},\mathbf{y},z) \vee \psi_2(\mathbf{x},\mathbf{y},z)$, where ψ_1 and ψ_2 are exclusive. So we have that $r_0 = s_0 + t_0$ where $s_0 = \lim_n \mu_n(\psi_1(\mathbf{c},\mathbf{d},e))$ and $t_0 = \lim_n \mu_n(\psi_2(\mathbf{c},\mathbf{d},e))$.

Writing $r = s + t$, where $s < s_0$ and $t < t_0$, we can see that the implication

$$(\exists^{\geq s}\mathbf{y}z)\psi_1(\mathbf{x},\mathbf{y},z) \wedge (\exists^{\geq t}\mathbf{y}z)\psi_2(\mathbf{x},\mathbf{y},z) \longrightarrow (\exists^{\geq r}\mathbf{y}z)\psi(\mathbf{x},\mathbf{y},z)$$

is logically valid.

Thus if both $(\exists^{\geq s}\mathbf{y}z)\psi_1(\mathbf{x},\mathbf{y},z) \equiv \mathbf{T}$ a.s. and $(\exists^{\geq t}\mathbf{y}z)\psi_2(\mathbf{x},\mathbf{y},z) \equiv \mathbf{T}$ a.s., then $(\exists^{\geq r}\mathbf{y}z)\psi(\mathbf{x},\mathbf{y},z) \equiv \mathbf{T}$ a.s..

So it's enough to prove Case 1 for the disjuncts, i.e. without loss of generality we assume that $\psi(\mathbf{x},\mathbf{y},z)$ is a conjunction of atomic and negation of atomic formulas, and let's write it as:

$$\psi(\mathbf{x},\mathbf{y},z) = \bigwedge_{i=1}^{m} \alpha_i(\mathbf{x},\mathbf{y},z) \wedge \bigwedge_{i=1}^{k} \beta_i(\mathbf{x},\mathbf{y}),$$

where z appears in each $\alpha_i(\mathbf{x},\mathbf{y},z)$.

From the form of $\psi(\mathbf{x},\mathbf{y},z)$ we can see that $r_0 = 1/2^{m+k}$. Thus we can write $r = st$, where $s < 1/2^m$ and $t < 1/2^k$. Now since the implication

$$(\exists^{\geq t}\mathbf{y})(\exists^{\geq s}z)\psi(\mathbf{x},\mathbf{y},z) \longrightarrow (\exists^{\geq st}\mathbf{y}z)\psi(\mathbf{x},\mathbf{y},z)$$

is logically valid, we just need to prove that $(\exists^{\geq t}\mathbf{y})(\exists^{\geq s}z)\psi(\mathbf{x},\mathbf{y},z) \equiv \mathbf{T}$ almost surely. But we have that:

$$\begin{aligned}
(\exists^{\geq t}\mathbf{y})(\exists^{\geq s}z)\psi(\mathbf{x},\mathbf{y},z) &= (\exists^{\geq t}\mathbf{y})(\exists^{\geq s}z)\left(\bigwedge_{i=1}^{m} \alpha_i(\mathbf{x},\mathbf{y},z) \wedge \bigwedge_{i=1}^{k} \beta_i(\mathbf{x},\mathbf{y})\right) \\
&\equiv (\exists^{\geq t}\mathbf{y})\left(\left((\exists^{\geq s}z)\bigwedge_{i=1}^{m} \alpha_i(\mathbf{x},\mathbf{y},z)\right) \wedge \bigwedge_{i=1}^{k} \beta_i(\mathbf{x},\mathbf{y})\right) \\
&\equiv (\exists^{\geq t}\mathbf{y})\bigwedge_{i=1}^{k} \beta_i(\mathbf{x},\mathbf{y}) \text{ a.s.} \\
&\equiv \mathbf{T} \text{ a.s.,}
\end{aligned}$$

where we used the induction hypothesis in the last two a.s. equivalences, since $s < 1/2^m = \lim_n \mu_n\left(\bigwedge_{i=1}^{m} \alpha_i(\mathbf{c},\mathbf{d},e)\right)$ and $t < 1/2^k = \lim_n \mu_n\left(\bigwedge_{i=1}^{k} \beta_i(\mathbf{c},\mathbf{d})\right)$.

Case 2: $r > r_0 = \lim_n \mu_n(\psi(\mathbf{c},\mathbf{d},e))$.

So $1-r < 1-r_0 = \lim_n \mu_n(\neg\psi(\mathbf{c},\mathbf{d},e))$. Thus, applying Case 1 on the formula $\neg\psi(\mathbf{x},\mathbf{y},z)$, we see that $(\exists^{>1-r}\mathbf{y}z)\neg\psi(\mathbf{x},\mathbf{y},z) \equiv \mathbf{T}$ a.s. (since the inequality $(> 1-r)$ can be treated like $(\geq 1-r)$).

So its negation $(\exists^{\geq r}\mathbf{y}z)\psi(\mathbf{x},\mathbf{y},z) \equiv \mathbf{F}$ a.s.. ∎

Using this lemma together with Lemma 8.9, Theorem 8.10 can be strengthened to:

Theorem 8.13 *With respect to the uniform measure, $\mathcal{L}^-_{\omega P}(\mathbf{x}) \leq_{w.a.s.} \mathbf{FO_0}(\mathbf{x})$.*

Proof: We just duplicate the proof of Theorem 8.10 except that in the case $\phi(\mathbf{x})$ is of the form $(\exists^{\geq r}\mathbf{y})\psi(\mathbf{x},\mathbf{y})$ we use the induction hypothesis together with the restriction we have on the formulas of $\mathcal{L}^-_{\omega P}$ and apply Lemma 8.12. ∎

As a corollary, taking $\mathbf{x}$ to be empty, we now fulfill our promise in Section 6.1.

Corollary 8.14 *For the uniform measure, the logic $\mathcal{L}^-_{\omega P}$ has the strong 0-1 law.*

∎

Now we're ready to prove the strong convergence law.

Theorem 8.15 *For the uniform measure, the logic $\mathcal{L}^-_{\omega P}$ has the strong convergence law for formulas. Moreover, for each formula $\phi(\mathbf{x})$ in that logic, $\mu(\underline{\phi(\mathbf{x})}) = \mu(\overline{\phi(\mathbf{x})})$ always takes values of the form $l/2^s$, where l and s are positive integers determined by the form of $\phi(\mathbf{x})$.*

Proof: Using Theorem 8.13 and Proposition 8.7, all we need to prove is that $\mathbf{FO_0}(\mathbf{x})$ has the strong convergence law. So let $\phi(\mathbf{x})$ be a quantifier free formula, and let $r_0 = \lim_n \mu_n(\phi(\mathbf{c}))$. Then r_0 will be of the form $l/2^s$. From the remarks after Corollary 8.6, we just need to show that for every $\delta > 0$, we have:

$$\mu\left(\overline{(\exists^{>r_0+\delta}\mathbf{x})\phi(\mathbf{x})}\right) = 0 \text{ and } \mu\left(\underline{(\exists^{\geq r_0-\delta}\mathbf{x})\phi(\mathbf{x})}\right) = 1.$$

But this is evident from Lemma 8.12. ■

Since the first order logic **FO** is a subset of the noncritical probability logic $\mathcal{L}^-_{\omega P}$, and from Proposition 8.7 and Theorem 5.6 we get:

Corollary 8.16 *For the uniform measure, the logics* **FO***,* **IFP***,* **PFP***, and* $\mathcal{L}^\omega_{\infty\omega}$ *have the strong convergence law for formulas.*

■

Chapter 9

Other strong convergence laws.

In this chapter we strengthen some of the theorems of Chapter 6 to prove results about the strong convergence for formulas. Let's first start with a useful tool.

Proposition 9.1 *If for some measure* μ *and some abstract formulas* $\phi(\mathbf{x}), \psi(\mathbf{x})$ *we have*

$$\models (\forall \mathbf{x})(\phi(\mathbf{x}) \to \psi(\mathbf{x})),$$

then

$$\mu(\underline{\phi(\mathbf{x})}) \leq \mu(\underline{\psi(\mathbf{x})}) \text{ and } \mu(\overline{\phi(\mathbf{x})}) \leq \mu(\overline{\psi(\mathbf{x})}).$$

In particular,

$$\text{if } \mu(\underline{\phi(\mathbf{x})}) = 1, \text{ then } \mu(\underline{\psi(\mathbf{x})}) = 1.$$

■

Corollary 9.2 *If* $\mu(\underline{\phi(\mathbf{x})}) = 1$ *and for some formulas* $\psi(\mathbf{x}), \psi'(\mathbf{x})$ *we have*

$$\models (\forall \mathbf{x})(\phi(\mathbf{x}) \to (\psi(\mathbf{x}) \leftrightarrow \psi'(\mathbf{x}))),$$

then $\mu(\underline{\psi(\mathbf{x})}) = \mu(\underline{\psi'(\mathbf{x})})$ *and* $\mu(\overline{\psi(\mathbf{x})}) = \mu(\overline{\psi'(\mathbf{x})})$.

In particular, if $\psi'(\mathbf{x})$ *has strong convergence, then so also does* $\psi(\mathbf{x})$.

■

9.1 Sparse random graphs.

In Section 6.4 we investigated the presence of the strong 0-1 law in the class of graphs, with the measure μ_n induced by independent Bernoulli trials with probability $p(n)$ for each edge.

The situation for the strong convergence law is slightly different.

Theorem 9.3 *Let the measure μ_n be induced by independent edge probability $p(n)$, and let $q(n)$ denote $(1 - p(n))$. Then a strong convergence law for formulas holds iff either*

$(\mathbf{0}) : \sum_n n^2 p(n) < \infty$, *or*

$(\overline{\mathbf{0}}) : \sum_n n^2 q(n) < \infty$, *or*

$(\mathbf{1}') :$ *For every $\epsilon > 0$, both $p(n), q(n) > 1/n^\epsilon$ for sufficiently large n, and $\lim_n p(n)$ exists.*

Proof: Since the strong convergence law is stronger than the strong 0-1 law, using Theorem 6.9 we only need to prove that each of the statements $(\mathbf{0})$,$(\overline{\mathbf{0}})$,$(\mathbf{1}')$ implies the strong convergence law, and the nonexistence of $\lim_n p(n)$ violates it.

Starting with $(\mathbf{0})$, we know from the proof of Theorem 6.9 that the sentence $\neg\phi_{K_2}$ "The graph is empty" has strong measure 1. Thus $\mathcal{A}_n$ will become empty eventually almost surely. But then, any formula $\psi(\mathbf{x})$ with the free variables $\mathbf{x} = \langle x_1, .., x_k \rangle$ can be reduced almost surely to a formula $\theta(\mathbf{x})$ in just the language of equality. Using Ehrenfeucht-Fraïssé games or by quantifier elimination one can show that for sufficiently big models, $\theta(\mathbf{x})$ is equivalent to a quantifier-free formula, all of whose atomic parts are equations between elements of $\mathbf{x}$.

Let $\phi(\mathbf{x})$ be the conjunction of the inequalities $(x_i \neq x_j)$ for $1 \leq i < j \leq k$. We can check that $\mu(\underline{\phi(\mathbf{x})}) = 1$. Also we have that either

$$\models (\forall \mathbf{x})(\phi(\mathbf{x}) \rightarrow \theta(\mathbf{x})) \text{ or } \models (\forall \mathbf{x})(\phi(\mathbf{x}) \rightarrow \neg\theta(\mathbf{x})).$$

Thus either

$$\mu(\underline{\psi(\mathbf{x})}) = \mu(\underline{\theta(\mathbf{x})}) = 1 \text{ or } \mu(\overline{\psi(\mathbf{x})}) = \mu(\overline{\theta(\mathbf{x})}) = 0,$$

and we have a strong convergence (to either 1 or 0).

Case $(\overline{\mathbf{0}})$ is similar as $\mathcal{A}_n$ will become the complete graph eventually almost surely.

It remains to consider $(\mathbf{1}')$. Working first with the uniform measure u, we know that every first order formula $\psi(\mathbf{x})$ collapses to a quantifier-free formula $\theta(\mathbf{x})$ almost surely, i.e.

$$u\left(\underline{(\forall \mathbf{x})(\psi(\mathbf{x}) \leftrightarrow \theta(\mathbf{x}))}\right) = 1.$$

Since the random theory Ψ of all extension axioms is the theory of all sentences of strong uniform measure 1, we have that:

$$\Psi \models (\forall \mathbf{x})(\psi(\mathbf{x}) \leftrightarrow \theta(\mathbf{x})).$$

Using compactness we get a finite subset $\Psi_0 \subset \Psi$, such that

$$\Psi_0 \models (\forall \mathbf{x})(\psi(\mathbf{x}) \leftrightarrow \theta(\mathbf{x})).$$

From the proof of Theorem 6.9 we know that each extension axiom $\psi \in \Psi_0$ will have strong measure 1, and thus with respect to the measure induced by $p(n)$, $\mathbf{FO}(\mathbf{x}) \leq_{w.a.s.} \mathbf{FO_0}(\mathbf{x})$.

Now if $\lim_n p(n)$ exists we can calculate $\lim_n \mu_n(\theta(\mathbf{c}))$ for each quantifier free $\theta(\mathbf{x})$. It's not hard to see that the proofs of Lemma 8.12 and Theorem 8.14 work, and we have the strong convergence law for $\mathbf{FO_0}(\mathbf{x})$, and consequently for $\mathbf{FO}(\mathbf{x})$.

However, if $\underline{\lim}_n p(n) < \overline{\lim}_n p(n)$ then we can check that for the atomic formula Rxy we have:

$$\mu(\underline{Rxy}) = \underline{\lim}_n p(n) < \overline{\lim}_n p(n) = \mu(\overline{Rxy}),$$

violating the strong convergence law. ∎

9.2 Sparse unary predicates.

We now investigate the presence of the strong convergence law for the class of cycles with random unary predicates (holding independently with probability $p(n)$), which was considered in Section 6.3.

As in Sections 6.3 and 6.4, we note a similarity between the sparse random graphs and the cycles with sparse unary predicates. However, the case of cycles is slightly more involved. We start with some definitions.

Definition 9.4 *Let the formula $C(x_1, .., x_k)$ denote the conjunction of the atomic formulas $C(x_1, x_i, x_j)$ for $1 < i < j \leq k$. Thus $C(x_1, .., x_k)$ says that starting with x_1 and going clockwise we meet the variables $x_2, .., x_k, x_1$ in the order described.*

We allow ourselves to insert a word w (over the alphabet $\{0, 1\}$) between two variables x_i, x_j and write $C(x_i, w, x_j)$ for the formula that says that between x_i and x_j we can find successive elements of the cycle with w coding the unary relation U on those elements.

For $m \geq 0$ we define the formulas $S_m(x,y)$ by induction on m:

$$S_0(x,y) = (x = y),\ \ S_1(x,y) = \neg(\exists z)C(x,z,y).$$

If m is even (say $m = 2l$, where $l \geq 1$), then

$$S_m(x,y) = (\exists z)(S_l(x,z) \wedge S_l(z,y)).$$

If m is odd (say $m = 2l+1$, where $l \geq 1$), then

$$S_m(x,y) = (\exists z)(S_{l+1}(x,z) \wedge S_l(z,y)).$$

Thus $S_m(x,y)$, having a quantifier depth $= \lceil \log_2 m \rceil + 1$, says that starting from x and going m steps clockwise we meet y.

An m-local formula $\eta(x)$ is a formula that talks only about elements on the cycle within distance $\leq m$ from x, that is, elements y for which either $S_l(x,y)$ or $S_l(y,x)$ holds for some $l \leq m$.

Note that we can describe any m-neighborhood of a variable x by a local formula $\eta(x)$ of quantifier depth $= \lceil \log_2 m \rceil + 1$.

Theorem 9.5 *In the class of random unary predicates with the ternary cyclic relation, let $p(n)$ denote the independent probabilities of $U(i)$ for $i \in \{1,..,n\}$, and let $q(n) = 1 - p(n)$. Then a strong convergence law for formulas holds iff either*

$(\mathbf{0}) : \sum_n np(n) < \infty$, *or*

$(\overline{\mathbf{0}}) : \sum_n nq(n) < \infty$, *or*

$(\mathbf{1}') :$ *For every $\epsilon > 0$, both $p(n), q(n) > 1/n^{\epsilon}$ for sufficiently large n, and $\lim_n p(n)$ exists.*

Proof: Again, using Theorem 6.4, we only need to show that each of the statements $(\mathbf{0})$,$(\overline{\mathbf{0}})$,$(\mathbf{1}')$ implies the strong convergence law, and the nonexistence of $\lim_n p(n)$ violates it.

Assuming $(\mathbf{0})$, we know from the proof of Theorem 6.4 that the sentence $\neg\phi_{w_1}$ "The unary predicate is empty" has strong measure 1. Thus $\mathcal{A}_n$ will become a naked cycle eventually almost surely.

Thus any formula $\psi(\mathbf{x})$ reduces almost surely to a formula in just the language of equality and the cyclic relation C. So without loss of generality we assume that $\psi(\mathbf{x})$ does not contain the unary predicates U.

Let $\mathbf{x} = \langle x_1, .., x_k\rangle$ and let t be the quantifier depth of $\psi(\mathbf{x})$. Using Ehrenfeucht-Fraïssé games one can show that for sufficiently big models, $\psi(\mathbf{x})$ is equivalent to a Boolean combination of formulas of the form

$$C(x_1, x_{\pi(2)}, .., x_{\pi(k)}),$$

where π is a permutation of the set $\{2, .., k\}$ or

$$S_m(x_i, x_j),$$

where $1 \leq i, j \leq k$ and $0 \leq m < 2^{t-1}$. (As in Section 6.3, Duplicator wins the t-move game by preserving the cyclic order of the elements chosen and distances $< 2^{t-i}$ at move i).

However, in a big model "most" interpretations of the tuple $\mathbf{x}$ will be "scattered". More precisely, since for a fixed m, $\neg S_m(x_i, x_j)$ will have "inner" measure 1, the formula

$$\phi(\mathbf{x}) = \bigwedge_{\substack{1\leq i,j\leq k,\\ 0\leq m<2^{t-1}}} \neg S_m(x_i, x_j)$$

will also have "inner" measure 1.

Thus, for sufficiently big models we have

$$\models (\forall \mathbf{x})(\phi(\mathbf{x}) \rightarrow (\psi(\mathbf{x}) \leftrightarrow \psi'(\mathbf{x}))),$$

where $\psi'(\mathbf{x})$ is a disjunction of exclusive cyclic expressions $C(x_1, x_{\pi(2)}, .., x_{\pi(k)})$.

Since, in a model of size $\geq k$, each $C(x_1, x_{\pi(2)}, .., x_{\pi(k)})$ will hold with probability $= 1/(k-1)!$, $\psi'(\mathbf{x})$ will hold with probability $= s/(k-1)!$, for some $s \in \{0, .., (k-1)!\}$.

By Corollary 9.2 we have that

$$\mu(\underline{\psi(\mathbf{x})}) = \mu(\overline{\psi(\mathbf{x})}) = s/(k-1)!,$$

and we have a strong convergence law.

Case ($\overline{\mathbf{0}}$) is similar as U will hold for all elements of the model eventually almost surely.

Now let's consider Statement ($\mathbf{1}'$). Here we want to express the fact that for a fixed quantifier depth t, in "most" big models "most" of the interpretations of the variables in $\mathbf{x}$ are scattered, and moreover, we can find between successive x_i, x_j a persistent word w of t. Thus we define:

$$\phi(\mathbf{x}) = \bigwedge_{1 \leq i,j \leq k} C(x_i, w, x_j).$$

Using the same arguments of the proof of Theorem 6.4, we can get that for each i, j, $\mu(\underline{C(x_i, w, x_j)}) = 1$, and thus $\mu(\underline{\phi(\mathbf{x})}) = 1$.

Also, using Ehrenfeucht-Fraïssé games, as in Case ($\mathbf{0}$) one can show that every formula $\psi(\mathbf{x})$ of quantifier depth t can be written as a Boolean combination of

(2^{t-1})-local formulas $\eta(x_i)$, the cyclic expressions $C(x_1, x_{\pi(2)}, .., x_{\pi(k)})$ (with π a permutation of the set $\{2, .., k\}$) and $S_m(x_i, x_j)$ (with $1 \leq i, j \leq k$ and $0 \leq m < 2^{t-1}$).

However, using the formula $\phi(\mathbf{x})$, we can get rid of all subformulas of the form $S_m(x_i, x_j)$ and get a formula $\psi_0(\mathbf{x})$, which is a Boolean combination of just the (2^{t-1})-local formulas $\eta(x_i)$ and the cyclic expressions $C(x_1, x_{\pi(2)}, .., x_{\pi(k)})$, such that:

$$\models (\forall \mathbf{x})(\phi(\mathbf{x}) \rightarrow (\psi(\mathbf{x}) \leftrightarrow \psi_0(\mathbf{x}))).$$

(Note that for $0 \leq m < 2^{t-1}$, $S_m(x_i, x_j)$ is refuted by $\phi(\mathbf{x})$, since a persistent word w of t is easily seen to have length $\geq 2^{t-1}$).

It then remains to show that $\psi_0(\mathbf{x})$ has strong convergence, since then by Corollary 9.2 $\psi(\mathbf{x})$ also has strong convergence.

Let's write the formula $\psi_0(\mathbf{x})$ in a full disjunctive normal form, so that a typical (exclusive) disjunct will be of the form

$$\theta(\mathbf{x}) = C(x_1, x_{\pi(2)}, .., x_{\pi(k)}) \wedge \bigwedge_{i=1}^{k} \eta_i(x_i),$$

where π is a permutation of the set $\{2, .., k\}$, and the formulas $\eta_i(x_i)$ are (2^{t-1})-local. From the proof of Lemma 8.12 we only have to prove the strong convergence for $\theta(\mathbf{x})$.

We fix a $\delta > 0$. For each n, we divide the n-cycle into $\sqrt{n}$ blocks of length $\sqrt{n}$ (rounding off as necessary). We say that a block is *good* if the probability of each $\eta_i(x_i)$ for x_i in the block is within δ of its expected probability.

Using the fact that $\lim_n p(n)$ exists, the Strong Law of Large Numbers guarantees that all but at most $\delta\sqrt{n}$ blocks in A_n are good eventually almost surely.

Now estimate the probability of $\theta(\mathbf{x})$ in A_n by first choosing a tuple of blocks B_i and then independently choosing an $x_i \in B_i$ for each i.

We can easily see that the probability of the blocks being distinct and good, and the x_i are at least the required distance from the block boundaries, is $\geq (1-\delta)$ eventually almost surely.

From the independence among the events that the blocks B_i are in the right order and each $\eta_i(x_i)$ is satisfied in its assigned block, it follows that the probability of $\theta(\mathbf{x})$ in A_n is within δ of the product of the expected probabilities of its conjuncts.

Since δ is arbitrary we get the strong convergence for $\theta(\mathbf{x})$, showing that $(\mathbf{1}')$ implies the strong convergence.

If, however, $\underline{\lim}_n p(n) < \overline{\lim}_n p(n)$, as in the case of random graphs we can check that for the atomic formula $U(x)$ we have:

$$\mu(\underline{U(x)}) = \underline{\lim}_n p(n) < \overline{\lim}_n p(n) = \mu(\overline{U(x)}),$$

violating the strong convergence law. ∎

Chapter 10

Conclusion and open problems.

We set up a new framework for asymptotic probabilities, that has a ($\sigma-$additive) measure, through which we were able to define a strong 0-1 law. By developing a theory that is parallel to that of the 0-1 law, we saw that the strong 0-1 law holds in many cases where a 0-1 law holds.

However, there were some cases in which the 0-1 law held while the strong law didn't. This showed that we were dealing with a different animal.

In the cases where a 0-1 law holds, inspecting the old proofs can give us easy proofs of the strong law. But sometimes the old proofs seem to resist any strengthening and we either can find new strong proofs or we are able to find some counterexamples.

In [4] K. Compton proved the 0-1 law for the first order logic as well as the inductive fixed-point logic if the underlying class is the class of partial orders. In his proof he relied on a result by Kleitman and Rothschild [12], where they proved that the class of partial orders $\mathbf{C}$ can be exhausted by a class of "special" partial orders $\mathbf{D}$. Precisely stated they showed that $1 - d_n/c_n \in O(1/n)$, where $c_n = |\mathbf{C} \cap \mathbf{M}_n|$ and $d_n = |\mathbf{D} \cap \mathbf{M}_n|$.

It's not hard to see that this estimate is not enough for the strong 0-1 law. So it's interesting to see if there is an estimate that is better than Kleitman and

Rothschild's. If not, then can Compton's proof be modified, say by dealing with a "special" class other than **D**, to be able to prove the strong 0-1 law?

We also introduced a new notion of strong convergence law for formulas based on our framework, and were able to prove the strong convergence laws for some classes and measures for which the strong 0-1 law held.

In the course of those proofs we brought into attention the notion of almost sure quantifier elimination. This notion enabled us to formalize an elegant version of the classical proof of the 0-1 law found in [7]. Our version ties together the connection between the two proofs of the 0-1 law (by Glebskii et. al. and Fagin) on one side and the relation between those proofs and the classical quantifier elimination method in model theory texts, e.g. [1], on the other side.

However, it's not clear whether the strong convergence law for formulas holds for Compton's slow growing classes. A proof of such result may have to discover the combinatorial distribution of a variable tuple **x** among the connected components of big models.

Also it's an interesting problem to find a natural measure space to interpret the "inner" and "outer" measures $\mu(\underline{\phi(\mathbf{x})})$ and $\mu(\overline{\phi(\mathbf{x})})$. We leave this task to an enthusiastic reader.

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