The logic of tied implications, part 2: Syntax

Nehad N. Morsi\textsuperscript{a,}*, W. Lotfallah\textsuperscript{b}, M.S. El-Zekey\textsuperscript{c}

\textsuperscript{a}Department of Basic Sciences, Arab Academy for Science, Technology \& Maritime Transport, P.O. Box 2033 Al-Horraya, Heliopolis, Cairo, Egypt
\textsuperscript{b}Department of Engineering Mathematics and Physics, Faculty of Engineering, Cairo University, Giza, Egypt
\textsuperscript{c}Department of Basic Sciences, Benha High Institute of Technology, Benha, Egypt

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Abstract

An implication operator $A$ is said to be tied if there is a binary operation $T$ that ties $A$; that is, the identity $A(a, A(b, z)) = A(T(a, b), z)$ holds for all $a, b, z$. We aim at the construction of a complete predicate logic for prelinear tied adjointness algebras. We realize this in three steps. In the first step, we establish a propositional calculus $\texttt{AdjTPC}$, complete for the class of all tied adjointness algebras on partially ordered sets; without prelinearity and ignoring the lattice operations. For that we supply a Hilbert system based on seven axioms and one deduction rule (modus ponens). In the second and third steps, we extend $\texttt{AdjTPC}$ to propositional and predicate calculi; complete for prelinear tied adjointness algebras. We apply a duality principle, due to Morsi, in all three calculi; through which we manage to cut down the number of proofs.

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1. Introduction

We continue the study of tied adjointness algebras, begun in [1,14] and Part 1 [15]. The implications $A$ of those algebras are tied by commutative integral ordered monoid operations $T$ in the following manner:

$\forall a, b, z : \ A(a, A(b, z)) = A(T(a, b), z).$

In Section 2, we present a complete syntax $\texttt{AdjTPC}$ for the class $\texttt{ADJT}$ of tied adjointness algebras, in the form of a Hilbert system. It features seven axioms and one inference rule, namely, the usual Modus Ponens rule of residuated logic. We deduce in Section 2.3 enough theorems and inferences in $\texttt{AdjTPC}$ to establish, in Section 2.4, its completeness for $\texttt{ADJT}$; by means of a quotient-algebra structure (a Lindenbaum type of algebra).

In Section 3, we construct a complete propositional calculus $\texttt{L-AdjTPC}$ for the class $\texttt{L-ADJT}$ of all prelinear tied adjointness algebras; as an enrichment of $\texttt{AdjTPC}$. The chain completeness of $\texttt{L-AdjTPC}$, with respect to all tied adjointness chains, is derived also through the representation theorem established in Part 1 [15].

* Corresponding author. Tel.: +20 2 4035460.
E-mail address: nchadmorsi@cairo.aast.edu (N.N. Morsi).

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Then in Section 4 we construct the predicate calculus extension $L$-$AdjTV$ of $L$-$AdjTPC$, whereby two quantifiers $\forall$ and $\exists$ are considered. We use Hájek’s notions of Henkin and linearly complete theories to establish the chain completeness of $L$-$AdjTV$, under the interpretations of $\forall$ and $\exists$ as inf and sup, respectively; in the manner of Hájek’s proof within $BL$ [8]. We also prove its completeness for the subclass $sHL$-$ADJT$, of algebras in $L$-$ADJT$ whose residuated parts are based on co-semi-Heyting algebras. In fact, we prove that both lattices underlying an algebra in $sHL$-$ADJT$ are always bi-semi-Heyting algebras.

We adapt a duality principle in $ADJT$, due to Morsi [14], to all three syntaxes. Through it we achieve a good reduction in the number of proofs.

Our conclusions are given summarily in Section 5.

2. The tied propositional calculus

2.1. Summary on tied adjointness algebras

We devote this subsection to a brief summary of the concepts and results of the first part [15] of this work.

Throughout, $(P, \leq_P)$ and $(L, \leq_L)$ denote partially ordered sets (posets). $P$ has a top element 1. But $L$ need not have a top element.

Definition 2.1.1 (Cf. Morsi [14]). An adjointness algebra is an 8-tuple

$$(L, \leq_L, P, \leq_P, 1, A, K, H),$$

in which $(L, \leq_L)$, $(P, \leq_P)$ are two posets with a top element 1 for $(P, \leq_P)$, and the following four conditions are satisfied:

(i) The implication $A : P \times L \to L$ is antitone in the left argument and monotone in the right argument, and it has $1 \in P$ as a left identity element.

(ii) The conjunction $K : P \times L \to L$ is monotone in both arguments and has $1 \in P$ as a left identity element.

(iii) The comparator $H : L \times L \to P$ is antitone in the left argument and monotone in the right argument, and it satisfies

$$\forall y, z \in L : H(y, z) = 1 \iff y \leq_L z.$$  \hspace{1cm} (1)

(iv) The three operations $A$, $K$ and $H$ are mutually related by the following condition, $\forall a \in P, \forall y, z \in L$:

$$\text{Adjointness : } y \leq_L A(a, z) \iff K(a, y) \leq_L z \iff a \leq_P H(y, z).$$

We call the ordered triple $(A, K, H)$ an implication triple on $(P, L)$. (In the case $P = L$, we say “on $P$” to mean “on $(P, P)$”.)

An adjointness lattice is an adjointness algebra whose two underlying posets are lattices. A complete adjointness lattice is one over two complete lattices. An adjointness chain is an adjointness lattice whose two orders are linear.

Definition 2.1.2. A residuated algebra is a special case of adjointness algebras over one poset, in which the conjunction is both commutative and associative. For a residuated algebra $(P, \leq_P, \leq_P, 1, I, T, I)$, we use the simpler notation $(P, \leq_P, 1, T, I)$. A residuated lattice $(P, \leq_P, \land, \lor, 1, T, I)$ is a residuated algebra whose underlying poset is a lattice with top element 1. A complete residuated lattice is a residuated algebra over a complete lattice. A residuated chain is a residuated lattice whose order is linear. We call such $T$ a triangular norm, slightly abusing this term, which is usually reserved for the case $P$ is the unit interval of real numbers only.

For a clear overview of t-norm based logical systems ranging from Monoidal t-norm logic to the well-known Łukasiewicz, Product and Gödel logics, the reader is referred to Gottwald’s book [7].

We use the term Residuation for the form taken by the condition Adjointness in the case of a residuated algebra $(P, \leq_P, 1, T, I)$. Thus, $(P, \leq_P, 1, T, I)$ has to satisfy: for all $a, b, c$ in $P$:

$$\text{Residuation : } T(a, b) \leq_P c \iff b \leq_P I(a, c) \iff a \leq_P I(b, c).$$  \hspace{1cm} (2)
We need the following identity satisfied [15] by all comparators $H$ in all adjointness algebras: for any subset \( \{z_j\}_{j \in \mathcal{J}} \) of $L$ that has a supremum in $L$, and any subset \( \{w_k\}_{k \in \Omega} \) of $L$ that has an infimum in $L$:

\[
H \left( \sup_{j \in \mathcal{J}} z_j, \inf_{k \in \Omega} w_k \right) = \inf_{j \in \mathcal{J}, k \in \Omega} H(z_j, w_k).
\]  

(3)

**Definition 2.1.3 (Morsi [14,15]).** A tied adjointness algebra is a mathematical system \((L, \preceq_L, P, \preceq_P, 1, A, K, H, T, I)\) in which \((L, \preceq_L, P, \preceq_P, 1, A, K, H)\) is an adjointness algebra, \((P, \preceq_P, 1, T, I)\) is a residuated algebra, and $T$ ties $A$; that is,

\[
\forall a, b \in P, \ \forall z \in L : \ A(T(a, b), z) = A(a, A(b, z)).
\]  

(4)

The class of all tied adjointness algebras is denoted by \text{ADJT}.

### 2.2. Language and axioms

We develop a new complete syntax for the semantics based on tied adjointness algebras. We call it \text{Propositional Calculus for Tied Implications}, and denote it by \text{AdjTPC}. The semantics of our logic assumes two partially ordered sets $L$ and $P$, whose elements are considered as \textit{truth-values}, as well as the five logical connectives of a tied adjointness algebra. Such a formal system can serve as a combined calculus for a pair of two, possibly different, types of uncertainty.

The language of \text{AdjTPC} consists of two denumerable sets $WF_L$ and $WF_P$ of formulae, and five logical connectives as follows: an \textit{implication} $\Rightarrow: WF_P \times WF_L \rightarrow WF_L$, a \textit{conjunction} $\&: WF_P \times WF_L \rightarrow WF_L$, a \textit{comparator} $\supset: WF_L \times WF_L \rightarrow WF_P$, a \textit{tying-conjunction} $\star: WF_P \times WF_P \rightarrow WF_P$, and an \textit{R-implication} $\rightarrow: WF_P \times WF_P \rightarrow WF_P$. The two sets $WF_L$ and $WF_P$ are constructed from two disjoint denumerable subsets $WF_{L_0}$ and $WF_{P_0}$ of atomic formulae, by means of repeated application of the logical connectives.

In an interpretation of \text{AdjTPC}, the five logical connectives $\Rightarrow$, $\&$, $\supset$, $\star$ and $\rightarrow$ will translate onto the five operations $A, K, H, T$ and $I$ of some tied adjointness algebra, respectively. Also, schemata of formulae (that is, formulae with metavariables denoting subformulae) will translate onto functions on truth values in $P \cup L$; built up as composites of $A, K, H, I$, and $T$. \text{AdjTPC} will be sound and complete for these semantics, in the sense that all theorems will translate onto all functions that are universally equal to 1.

The lowercase Greek letters $\alpha, \beta, \gamma, \delta$ are used as metavariables running on formulae in $WF_L$, while the lowercase Greek letters $\zeta, \xi, \tau, \eta, \chi$ are used as metavariables running on formulae in $WF_P$. We use the infix notation $\Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow 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These are the seven axioms we choose for \text{AdjTPC}. Their corresponding schemata of formulae in $WF_P$ are what follows:

- **Axioms of AdjTPC:**
  - **P 1** \( \xi \rightarrow (\tau \rightarrow \tau) \)
  - **P 2** \( (\xi \rightarrow (\tau \rightarrow \zeta)) \rightarrow ((\xi \star \tau) \rightarrow \zeta) \)
  - **P 3** \( ((\xi \star \tau) \rightarrow \eta) \rightarrow ((\eta \rightarrow \zeta) \rightarrow (\tau \rightarrow (\zeta \rightarrow \zeta))) \)
P 4 ⊢ ζ → (β ⊃ β)
P 5 ⊢ (ζ → (β ⊃ γ)) → ((ζ & β) ⊃ γ)
P 6 ⊢ ((ζ & β) ⊃ δ) → (δ ⊃ γ) → (β ⊃ (ζ ⇒ γ)))
P 7 ⊢ (β ⊃ (ζ ⇒ γ)) → (ζ → (β ⊃ γ)).

Inference Rule for AdjTPC:

MP: ζ, ξ → τ (ζ, τ ∈ WFₚ) (Modus Ponens for R-implication).

Note that the theorems of AdjTPC are certain formulae in WFₚ, but their subformulae are from WFₚ ∪ WFₖ.

Whereas WFₖ does not support a notion of absolute truth, and does not contain theorems. As such, MP always derives one formula in WFₚ from two formulae in WFₚ.

We comment on these axioms. Residuated logic is known to need the one inference rule MP only [11,16]. We point out that Morsi [13] has started from the three axioms P1, P2, and P3, together with MP, and developed a Residuated Propositional Calculus (RPC) with two logical connectives • and →, which stand for the two operations in a residuated algebra. We borrow these axioms from [13]. Axioms P4, P5 and P6 imitate P1, P2, and P3. We had to include also P7 in order to close the proof of adjoinness.

A theory Γ over AdjTPC is a set of formulae in WFₚ. Elements of Γ are called the special axioms of that theory. An inference (deduction, proof or derivation) Γ ⊢ τ in a theory Γ is a finite sequence τ₁, . . . , τₙ of formulae in WFₚ, whereby each τᵢ is either an axiom of AdjTPC, a member of Γ or follows from some preceding members of the sequence using MP, such that the last line of that sequence is τ. We also write Γ ⊢ A, where A is a nonempty finite subset of WFₚ, to abbreviate the writing of the set of deductions {Γ ⊢ τ | τ ∈ A}. When ∅ ⊢ τ (that is, τ is derived from the axioms of AdjTPC alone), we write ⊢ τ, and we call τ a theorem.

A considerable simplification of notation is achieved by using two new symbols “⊂⊃” and “←→”. The notation x ⊂⊃ y is used to abbreviate the writing of two formulae x ⊃ y and y ⊃ x. Thus, x ⊂⊃ y is a set of two formulae, and not one formula composed from two subformulae. Similarly, the notation x ←→ y is used to abbreviate the writing of two formulae x → y and y → x. The notation Γ ⊢ x ⊂⊃ y is an abbreviation of Γ ⊢ (x ⊃ y, y ⊃ x), and a theorem Γ ⊢ x ⊂⊃ y is an abbreviated writing of two theorems. Likewise for the notations Γ ⊢ x ←→ y and ⊢ x ←→ y. It will be proved that the two meta-predicates ⊢ x ⊂⊃ y on WFₚ, and ⊢ x ←→ y on WFₚ are equivalence relations, called equivalence in WFₖ, respectively in WFₚ. Another equivalence relation ≡ on WFₚ is defined by

ζ ≡ τ iff (ζ ⊢ τ and τ ⊢ ζ).

It follows from modus ponens that if ⊢ ζ ←→ τ then ζ ≡ τ, but not vice-versa (see Theorem 4.3.2). For instance, ζ ≡ ξ • ζ, but ξ ⊢ ζ → ξ • ζ.

To reduce the use of brackets, we maintain a convention of priority among the nine symbols ⇒, &, ⊃, ⊂⊃, ←→, ⊢, ≡, →, •. We give & and • the highest priority; whereas we give ⊢, ≡ less priority than the other symbols.

2.3. Theorems in AdjTPC

Morsi [13] has deduced from axioms P1, P2 and P3, by means of the deduction rule MP, a complete list of theorems in RPC. These remain valid in AdjTPC. Thus, we have

Theorem 2.3.1. The following are theorems in AdjTPC:

R 1 ⊢ (ζ • τ → ζ) → (ζ → (τ → ζ)),
R 2 ⊢ ζ → ζ (reflexivity of R-implication),
R 3 ⊢ ζ → (ζ → ζ),
R 4 ⊢ τ • ζ → ζ • τ (⋆ is commutative),
R 5 ⊢ ζ • (τ • ζ) ←→ (ζ • τ) • ζ (⋆ is associative),
R 6 ⊢ ζ • (τ • ζ) → τ • (ζ • ζ) (the exchange principle for ⋆),
R 7 ⊢ ζ • ζ → ζ, ⊢ ζ • ζ → ζ,
R 8 ⊢ (τ → ζ) • τ → ζ, ⊢ τ • (τ → ζ) → ζ (fuzzy modus ponens),
R 9 ⊢ (τ → (τ → ζ) → ζ),
R 10 ⊢ (ζ → (τ → ζ)) → (τ → (ζ → ζ)) (the exchange principle EP for →),
R 11 ⊢ τ → (ζ → ζ • τ), ⊢ τ → (ζ → ζ • τ),
R 12 ⊢ (ζ → τ) • (τ → ζ) → (ζ → ζ) (→ is fuzzy •-transitive),
R 13 ⊢ (ζ → χ) • (τ → ζ) → ((ζ → ζ) → (τ → ζ)),
R 14 ⊢ (ζ → χ) • (τ → ζ) → (ζ • τ → χ • ζ),
R 15 ⊢ (τ → ζ) → ((τ → χ) → (τ → ζ)),
R 16 ⊢ (τ → χ) → ((χ → ζ) → (τ → ζ)),
R 17 ⊢ (τ → ζ) → (ζ • τ → ζ • ζ),
R 18 ⊢ τ • (τ → ζ) → (ζ → ζ),
R 19 ⊢ ζ • (τ → ζ) → (τ → ζ • ζ),
R 20 ⊢ ζ • (τ → ζ) → ((ζ → τ) → ζ).

Also, the following deductions in RPC remain valid in AdjTPC (see [13] for proofs):

Proposition 2.3.1 (Transitivity of R-implication). ζ → τ, τ → ζ ⊢ ζ → ζ

The above inference and R2 demonstrate that the meta-predicate ⊢ ζ → ζ is a pre-order relation on WFp. Consequently, the meta-predicate ⊢ ζ ←→ ζ is an equivalence relation on WFp. We call it equivalent in WFp.

Proposition 2.3.2 (Residuation). τ → (ζ → ζ) ≡ ζ • τ → ζ ≡ ζ → (τ → ζ).

Proposition 2.3.3 (Monotonicity). The followings are correct inferences in AdjTPC:
ζ → τ ⊢ (τ → ζ) → (ζ → ζ),
ζ → τ ⊢ (ζ → ζ) → (ζ → τ),
ζ → τ ⊢ (ζ • ζ) → (ζ • ζ),
ζ → τ ⊢ (ζ • ζ) → (τ • ζ).

We next deduce new theorems, with the help of the other axioms P4–P7 as well.

Theorem 2.3.2 (Reflexivity of comparator). ⊢ β ⊳ β.

Proof. In P4, take for ζ one of the axioms, then use MP. □

Theorem 2.3.3. ⊢ (β ⊳ γ) & β ⊳ γ

Proof. Axiom P5 with ζ/β (i.e., with ζ replaced by β ⊳ γ) becomes ⊢ ((β ⊳ γ) → (β ⊳ γ)) → ((β ⊳ γ) & β ⊳ γ). Using R2 and MP, we get the result. □

Theorem 2.3.4. ⊢ (ζ → (β ⊳ γ)) ↔ (ζ & β ⊳ γ) ↔ (β ⊳ (ζ → γ)).

Proof. Axiom P6 with ζ/β becomes ((ζ & β ⊳ γ) → ((ζ & β ⊳ γ) → (β ⊳ (ζ → γ))), and so (ζ & β ⊳ γ) → (β ⊳ (ζ → γ)). We combine this with P5 and P7 to get the result. □

Applying MP on the preceding theorem we get the next proposition which corresponds to the condition Adjointness in the definition of adjointness algebras.

Proposition 2.3.4 (Adjointness). β ⊳ (ζ → γ) ≡ ζ & β ⊳ γ ≡ ζ → (β ⊳ γ).

Theorem 2.3.5.

\[ \vdash (δ ⊳ γ) → (β ⊳ ((β ⊳ δ) ⇒ γ)) \] (6)
\[ \vdash (δ ⊳ γ) & β ⊳ ((β ⊳ δ) ⇒ γ) \] (7)
\[ \vdash (δ ⊳ γ) → ((β ⊳ δ) → (β ⊳ γ)) \] (8)
\( \vdash (\beta \supset \delta) \rightarrow ((\delta \supset \gamma) \rightarrow (\beta \supset \gamma)) \),
\( (9) \)
\( \vdash (\beta \supset \delta) \star (\delta \supset \gamma) \rightarrow (\beta \supset \gamma) \) (\( \star \) is fuzzy \(-\)transitive),
\( (10) \)
\( \vdash (\alpha \supset \beta) \star (\gamma \supset \delta) \rightarrow ((\beta \supset \gamma) \rightarrow (\alpha \supset \delta)). \)
\( (11) \)

**Proof.** By P6 with \( \frac{\beta \supset \delta}{\xi} \), we have \( \vdash ((\beta \supset \delta) \& \beta \supset \delta) \rightarrow ((\delta \supset \gamma) \rightarrow (\beta \supset ((\beta \supset \delta) \Rightarrow \gamma))). \) Using this, Theorem 2.3.3 and MP, we get (6), and (7) follows by Adjointness.

By P7 we have \( \vdash (\beta \supset ((\beta \supset \delta) \Rightarrow \gamma)) \rightarrow ((\beta \supset \delta) \rightarrow (\beta \supset \gamma)). \) Combine with (6) to obtain (8). Then use Residuation for the next two formulae. By two applications of (10) we get \( \vdash (\alpha \supset \beta) \star (\gamma \supset \delta) \star (\beta \supset \gamma) \rightarrow (\alpha \supset \delta), \) from which (11) follows by Residuation. \( \square \)

**Proposition 2.3.5** (Transitivity of comparator). \( \beta \supset \delta, \delta \supset \gamma \vdash \beta \supset \gamma. \)

**Proof.** By Theorem 2.3.5 and MP. \( \square \)

Proposition 2.3.5 and Theorem 2.3.2 demonstrate that the meta-predicate \( \vdash \beta \supset \gamma \) is a pre-order relation on \( WF_L \). Therefore, the meta-predicate \( \vdash \beta \supset \supset \gamma \) is an equivalence relation on \( WF_L \). We call it equivalidity in \( WF_L \).

Throughout, we shall use the reflexivity of the two relations \( \xi \rightarrow \tau \) (on \( WF_P \)) and \( \vdash \beta \supset \gamma \) (on \( WF_L \)) (R2 and Theorem 2.3.2), usually as premises for MP, as matters of course without explicit mention of their use. Likewise, we shall say “combine” to indicate a use of the transitivity of either one of these two relations (Proposition 2.3.1, 2.3.5), which will be clear from the context.

**Theorem 2.3.6.** \( \vdash (\delta \supset (\xi \Rightarrow \beta)) \rightarrow ((\gamma \supset \delta) \rightarrow (\xi \& \gamma \supset \beta)). \)

**Proof.** As \( \vdash (\gamma \supset (\xi \Rightarrow \beta)) \rightarrow ((\xi \& \gamma \supset \beta) \) (Theorem 2.3.4), then by R15,
\( \vdash ((\gamma \supset \delta) \rightarrow (\gamma \supset (\xi \Rightarrow \beta))) \rightarrow ((\gamma \supset \delta) \rightarrow (\xi \& \gamma \supset \beta)) \).
But by (8), \( \vdash (\delta \supset (\xi \Rightarrow \beta)) \rightarrow ((\gamma \supset \delta) \rightarrow (\gamma \supset (\xi \Rightarrow \beta))). \)

Combine to get the assertion. \( \square \)

**Theorem 2.3.7.** \( \vdash \xi \rightarrow ((\xi \Rightarrow \gamma) \supset \gamma), \)
\( \vdash \xi \rightarrow (\beta \supset \xi \& \beta), \)
\( \vdash \xi \Rightarrow (\xi \Rightarrow \beta \supset \beta), \)
\( \vdash \xi \Rightarrow (\xi \Rightarrow \xi \& \beta), \)
\( \vdash \xi \& \beta \Rightarrow \gamma, \)
\( \vdash \xi \& \beta \Rightarrow \beta, \)
\( \vdash \gamma \supset (\xi \Rightarrow \gamma). \)

**Proof.** The first five formulae follow by applying Adjointness to the following three instances of Theorem 2.3.2 and R2: \( \vdash (\xi \Rightarrow \gamma) \supset (\xi \Rightarrow \gamma), \vdash \xi \Rightarrow \xi \& \beta \) and \( \vdash (\xi \Rightarrow \beta) \rightarrow (\xi \Rightarrow \beta). \)

The last two follow from P4 by Adjointness. \( \square \)

**Proposition 2.3.6** (Neutrality). \( \xi \vdash (\xi \Rightarrow \gamma) \supset \gamma, \)
\( \xi \vdash \xi \& \gamma \supset \gamma. \)

**Proof.** We deduce from Theorem 2.3.7, by hypothesis and MP, the formulae \( (\xi \Rightarrow \gamma) \supset \gamma \) and \( \gamma \supset \xi \& \gamma. \) This and Theorem 2.3.7 complete the proof. \( \square \)

**Lemma 2.3.1** (Duality (Cf. Morsi [14])). Let \( \Psi, \Phi \) be formulae in \( WF_P \) containing meta-variables as atomic formulae. Let \( \Psi^d, \Phi^d \) be the formulae obtained from \( \Psi, \Phi \) by interchanging the two connectives \( \Rightarrow, \& \), reversing the direction of the comparator \( \supset \) and keeping all other symbols fixed. Then in \( AdjTPC \), \( \Psi \vdash \Phi \) if and only if \( \Psi^d \vdash \Phi^d. \) In particular, \( \Phi \) is a theorem in \( AdjTPC \) if and only if \( \Phi^d \) is a theorem in \( AdjTPC. \)
Theorem 2.3.8. Theorem 2.3.4, Theorem 2.3.6, Theorem 2.3.4, whereas the inference rule (Duality being that the duals of those lines constitute a valid derivation of $\Phi^d$ from $\Psi^d$, because this duality is self-inverse, the four axioms $P_1, P_2, P_3, P_4$ are self-dual, and the duals of the remaining three axioms $P_5, P_6, P_7$ are, respectively, Theorem 2.3.4, Theorem 2.3.6, Theorem 2.3.4, whereas the inference rule MP is not affected by this duality. □

We shall use Duality in the sequel to cut down the number of proofs. Note that while the residuated logic RPC is a particularization of $\text{AdjTPC}$, and so Duality should hold there, it is very difficult to recognize this particular duality in RPC, because there $WF_P = WF_L, \rightarrow \Rightarrow \equiv$ and $\ast = \&$. On the other hand, this duality can be extended to a theory $\Gamma$ over $\text{AdjTPC}$ if and only if the duals of the special axioms of $\Gamma$ are theorems in $\Gamma$. This is the case for the theory of prelinear tied adjointness algebras, which is the subject of Sections 3 and 4.

Theorem 2.3.8.

\[ \vdash \xi \star (\beta \vdash \gamma) \rightarrow (\beta \vdash \xi \& \gamma), \quad (12) \]
\[ \vdash (\xi \rightarrow \tau) \& \beta \vdash (\xi \Rightarrow \tau \& \beta), \quad (13) \]
\[ \vdash \xi \star (\beta \vdash \gamma) \rightarrow ((\xi \Rightarrow \beta) \vdash \gamma), \quad (14) \]
\[ \vdash \xi \& (\tau \Rightarrow \beta) \vdash ((\xi \rightarrow \tau) \Rightarrow \beta), \quad (15) \]
\[ \vdash (\beta \vdash \gamma) \star (\xi \rightarrow \tau) \rightarrow ((\tau \Rightarrow \beta) \vdash (\xi \Rightarrow \gamma)), \quad (16) \]
\[ \vdash (\beta \vdash \gamma) \rightarrow ((\xi \Rightarrow \beta) \vdash (\xi \Rightarrow \gamma)), \quad (17) \]
\[ \vdash (\xi \rightarrow \tau) \rightarrow ((\tau \Rightarrow \gamma) \vdash (\xi \Rightarrow \gamma)), \quad (18) \]
\[ \vdash (\beta \vdash \gamma) \star (\xi \rightarrow \tau) \rightarrow ((\xi \& \beta \vdash (\tau \& \gamma)), \quad (19) \]
\[ \vdash (\beta \vdash \gamma) \rightarrow (\xi \& \beta \vdash \xi \& \gamma), \quad (20) \]
\[ \vdash (\xi \rightarrow \tau) \rightarrow (\xi \& \beta \vdash (\tau \& \beta)). \quad (21) \]

Proof. By Theorem 2.3.7, $\vdash \xi \rightarrow (\gamma \vdash \xi \& \gamma)$ Combine with Theorem 2.3.5: $\vdash \xi \rightarrow ((\beta \vdash \gamma) \rightarrow (\beta \vdash \xi \& \gamma))$, from which we get (12) by Residuation.

Next, as $\vdash \tau \rightarrow (\beta \vdash \tau \& \beta)$ (Theorem 2.3.7), then by R15
\[ \vdash (\xi \rightarrow \tau) \rightarrow (\xi \rightarrow (\beta \vdash \tau \& \beta)). \]

Also by Theorem 2.3.4, $\vdash (\xi \rightarrow (\beta \vdash \tau \& \beta)) \rightarrow (\beta \vdash (\xi \Rightarrow \tau \& \beta))$.

Combine then apply Adjointness to get (13).

Formulae (14), (15) now follow by Duality.

By (14), $\vdash (\beta \vdash \gamma) \star (\xi \rightarrow \tau) \rightarrow (((\xi \rightarrow \tau) \Rightarrow \beta) \vdash \gamma)$. Also, when we apply (9) to (15), $\vdash (((\xi \rightarrow \tau) \Rightarrow \beta) \vdash \gamma) \rightarrow ((\xi \& (\tau \Rightarrow \beta)) \vdash \gamma) \rightarrow ((\tau \Rightarrow \beta) \vdash (\xi \Rightarrow \gamma))$, by Theorem 2.3.4. Combine to deduce (16).

Formula (17) then ensues by taking $\tau = \xi$, and (18) by taking $\beta = \gamma$. The remaining three formulae follow by Duality. □

Applying MP on Theorems 2.3.5, 2.3.8, we obtain

Proposition 2.3.7 (Monotonicity). $\delta \supset \gamma \vdash (\beta \supset \delta) \rightarrow (\beta \supset \gamma)$,

$\beta \supset \delta \vdash (\delta \supset \gamma) \rightarrow (\beta \supset \gamma)$,

$\beta \supset \gamma \vdash (\xi \Rightarrow \beta) \vdash (\xi \Rightarrow \gamma)$,

$\xi \rightarrow \tau \vdash (\xi \& \beta \vdash (\tau \& \beta)$,

$\xi \rightarrow \tau \vdash (\tau \Rightarrow \gamma) \vdash (\xi \Rightarrow \gamma)$. 


Proposition 2.3.8 (Substitution Theorem).
\[
\tilde{\xi} \leftrightarrow \tau, \alpha \sqsupset \beta \vdash \Psi_L(\alpha, \tilde{\xi}) \sqsupset \Psi_L(\alpha/\beta, \tilde{\xi}/\tau),
\]
\[
\xi \leftrightarrow \tau, \alpha \sqsupset \beta \vdash \Phi_P(\alpha, \xi) \leftrightarrow \Phi_P(\alpha/\beta, \xi/\tau).
\]
where \(\Psi_L(\alpha, \tilde{\xi})\) is a formula in \(WF_L\) in which \(\alpha \in WF_L\) and \(\tilde{\xi} \in WF_P\) occur as subformulae, and \(\Psi_L(\alpha/\beta, \tilde{\xi}/\tau)\) is a formula obtained from \(\Psi_L(\alpha, \tilde{\xi})\) by substituting \(\beta\) for \(\alpha\) and \(\tau\) for \(\tilde{\xi}\); in one or more of the occurrences of \(\alpha\) and \(\tilde{\xi}\) in \(\Psi_L(\alpha, \tilde{\xi})\). Similarly for \(\Phi_P(\alpha, \xi)\) in \(WF_P\). In particular, substitution preserves both types of equivalidity.

Proof. This follows by complete induction on the complexity of a formula in \(WF_P \cup WF_L\), from the monotonicity Propositions 2.3.3 and 2.3.7 of the five logical connectives \(\rightarrow, \&\), \(\supset\), \(\rightarrow\) and \(*\).

Theorem 2.3.9.
\[
\vdash \xi \rightarrow ((\xi \rightarrow (\tau \rightarrow \beta)) \supset \beta),
\]
\[
\vdash \xi \rightarrow (\beta \supset \xi \& (\tau \& \beta)).
\]

Proof. By Theorem 2.3.7 we get \(\vdash \xi \rightarrow ((\xi \rightarrow (\tau \rightarrow \gamma)) \supset (\tau \rightarrow \gamma))\), and by Theorem 2.3.4 we get \(((\xi \rightarrow (\tau \rightarrow \gamma)) \supset (\tau \rightarrow \gamma)) \leftrightarrow (\tau \rightarrow ((\xi \rightarrow (\tau \rightarrow \gamma)) \supset \gamma))\). By combining we get \(\vdash \xi \rightarrow (\tau \rightarrow ((\xi \rightarrow (\tau \rightarrow \gamma)) \supset \gamma))\), and (22) follows by Residuation, whence (23) follows by Duality.

Theorem 2.3.10 (* ties \(\Rightarrow\) and \&).
\[
\vdash (\xi \& \tau \rightarrow \gamma) \sqsupset (\xi \rightarrow (\tau \rightarrow \gamma)),
\]
\[
\vdash (\xi \& \tau) \& \gamma \sqsupset \xi \& (\tau \& \gamma).
\]

Proof. By R11, \(\vdash \xi \rightarrow (\tau \rightarrow \xi \& \tau)\), and by Theorem 2.3.8, \(\vdash (\tau \rightarrow \xi \& \tau) \rightarrow ((\xi \& \tau \rightarrow \gamma) \supset (\tau \rightarrow \gamma))\). By combining the last two facts, we get \(\vdash \xi \rightarrow ((\xi \& \tau \rightarrow \gamma) \supset (\tau \rightarrow \gamma))\), from which we get by Adjointness \(\vdash (\xi \& \tau \rightarrow \gamma) \supset (\xi \rightarrow (\tau \rightarrow \gamma))\).

The other direction follows from (22) by Adjointness. This completes the proof of (24), and (25) follows by Duality.

Theorem 2.3.11 (Exchange principle).
\[
\vdash (\xi \rightarrow (\tau \rightarrow \gamma)) \supset (\tau \rightarrow (\xi \rightarrow \gamma)),
\]
\[
\vdash \xi \& (\tau \& \gamma) \supset \tau \& (\xi \& \gamma).
\]

Proof. By (24) and the commutativity of \(*\) (R4), we have the following equivalidities: \((\xi \rightarrow (\tau \rightarrow \gamma)) \supset (\xi \& \tau \rightarrow \gamma) \supset (\tau \& \xi \rightarrow \gamma) \supset (\tau \rightarrow (\xi \rightarrow \gamma))\).

The second formula follows by Duality.

Theorem 2.3.12 (Balance).
\[
\vdash (\alpha \supset \beta) \rightarrow (((\alpha \supset \gamma) \supset \delta) \supset ((\beta \supset \gamma) \supset \delta)),
\]
\[
\vdash (\alpha \supset \beta) \rightarrow (((\gamma \supset \alpha) \& \delta) \supset ((\gamma \supset \beta) \& \delta)),
\]
\[
\vdash (((\xi \rightarrow (\tau \rightarrow \beta)) \supset \beta) \rightarrow \gamma) \supset ((\xi \rightarrow (\tau \rightarrow \gamma)),
\]
\[
\vdash \xi \& (\tau \& \gamma) \supset (\beta \supset \xi \& (\tau \& \beta)) \& \gamma.
\]

Proof. By Theorem 2.3.5 then Theorem 2.3.8, \(\vdash (\alpha \supset \beta) \rightarrow ((\beta \supset \gamma) \rightarrow (\alpha \supset \gamma))\) and \(\vdash ((\beta \supset \gamma) \rightarrow (\alpha \supset \gamma)) \rightarrow (((\alpha \supset \gamma) \supset \delta) \supset ((\beta \supset \gamma) \supset \delta)).\) Combining we get (28), and (29) follows by Duality.
Next, by Theorem 2.3.9 and Proposition 2.3.7,
\[ \vdash (((\xi \Rightarrow (\tau \Rightarrow \beta)) \supset \beta) \Rightarrow \gamma) \supset ((\xi \Rightarrow \tau) \Rightarrow \gamma). \]
Combine this with (24) to get (30), from which (31) follows by Duality.  

The importance of these four formulae, called collectively balance, is that none of them has so far been proved for any adjointness algebra that is not tied, although the connectives $\star$ and $\rightarrow$ do not appear in either (30) or (31), and $\rightarrow$ is employed in (28) and (29) for ordering only. For further discussions on this assertion, see [1, Section 3.4].

**Theorem 2.3.13.** $\vdash (((\beta \supset \gamma) \supset \gamma) \supset \gamma) \iff (\beta \supset \gamma)$.

$\vdash (\beta \supset (\beta \supset \gamma) \& \beta) \iff (\beta \supset \gamma)$,

$\vdash (((\xi \Rightarrow (\gamma \Rightarrow \gamma)) \Rightarrow \gamma) \supset (\xi \Rightarrow \gamma)$,

$\vdash (\beta \supset \xi \& \beta) \supset \xi \& \beta$,

$\vdash (\xi \Rightarrow \xi \& \beta) \supset \xi \& \beta$,

$\vdash (\xi \Rightarrow (\xi \Rightarrow (\gamma \Rightarrow \beta))) \supset (\xi \Rightarrow \gamma)$.

**Proof.** By Theorem 2.3.7, $\vdash (\beta \supset (\beta \supset \gamma) \Rightarrow \gamma)$. So by Proposition 2.3.7, $\vdash (((\beta \supset \gamma) \supset \gamma) \Rightarrow (\beta \supset \gamma)$. This and Theorem 2.3.7 complete the proof of the first equivalence. The other five equivalences are proved similarly.  

The following interesting theorems are also easy to derive. Note that the first theorem is self-dual. The others are listed in dual pairs.

**Theorem 2.3.14.** $\vdash \xi \& (\tau \Rightarrow \gamma) \supset (\tau \Rightarrow \xi \& \gamma)$,

$\vdash (\tau \Rightarrow \gamma) \supset (\xi \& \tau \Rightarrow \xi \& \gamma)$,

$\vdash (\xi \Rightarrow (\xi \Rightarrow \gamma) \supset \tau \& \gamma$,

$\vdash \xi \& \beta \supset (\tau \Rightarrow (\xi \Rightarrow \tau) \& \beta)$,

$\vdash \xi \& (\xi \Rightarrow (\gamma \Rightarrow \beta)) \supset (\tau \Rightarrow \gamma)$,

$\vdash (\xi \& \beta \Rightarrow \gamma) \Rightarrow (\xi \Rightarrow \gamma)$.

A formal system (complete syntax), for adjointness algebras over one poset (that is, in the case $(L, \leq_L) = (P, \leq_P)$) has been developed in [12]. The resulting logic is called Propositional Calculus under Adjointness, abbreviated $\text{AdjTPC}$. It features seven axioms and four inference rules. If, in our system $\text{AdjTPC}$, we let the two sets $WF_L$ and $WF_P$ of formulae coincide, and adopt, besides $\text{MP}$, the following inference rule “$\beta \supset \gamma \equiv \beta \Rightarrow \gamma$”, we shall have a version of $\text{AdjTPC}$ for tied adjointness algebras over one poset. We leave to the reader the easy verification that all axioms and inference rules of $\text{AdjTPC}$ become theorems and deductions of this version of $\text{AdjTPC}$. But not vice-versa, not only because $\star$ and $\rightarrow$ are absent from $\text{AdjTPC}$, but also because some theorems of $\text{AdjTPC}$ fail in $\text{AdjPC}$ although $\star$ and $\rightarrow$ do not feature in them, for instance Theorem 2.3.11, (30) and (31). For algebraic counterexamples, we recall that commutative nonassociative conjunctions cannot be balanced [1, Theorem 3.2].

2.4. Semantics

We explain how $\text{ADJT}$ constitutes a semantic domain for $\text{AdjTPC}$. We prove that the syntax of $\text{AdjTPC}$ is sound for $\text{ADJT}$. We then construct a special model for $\text{AdjTPC}$, and we use it to prove completeness.

An interpretation of $\text{AdjTPC}$ is a triple $\mathfrak{I} = (A, \pi_L, \pi_P)$ in which $A = (L, \leq_L, P, \leq_P, 1, A, K, H, T, I)$ is a tied adjointness algebra, $\pi_L : WF_L \rightarrow L$ and $\pi_P : WF_P \rightarrow P$ are functions, called the valuation functions (or truth functions) of the interpretation; subject to the condition that the following five identities hold for all formulae $\xi, \tau \in WF_P$, and $\beta, \gamma \in WF_L$:

\[
\pi_L(\xi \Rightarrow \gamma) = A(\pi_P(\xi), \pi_L(\gamma)), \tag{32}
\]

\[
\pi_L(\xi \& \beta) = K(\pi_P(\xi), \pi_L(\beta)), \tag{33}
\]

\[
\pi_P(\beta \supset \gamma) = H(\pi_L(\beta), \pi_L(\gamma)), \tag{34}
\]
\[\pi_p(\xi \star \tau) = T(\pi_p(\xi), \pi_p(\tau)),\]  
\[\pi_p(\xi \rightarrow \tau) = I(\pi_p(\xi), \pi_p(\tau)).\]  

These conditions mean that the truth functions \(\pi_L\) and \(\pi_P\) translate the five logical connectives \(\Rightarrow, \&\), \(\lor\), \(\ast\) and \(\rightarrow\) (on \(WF_L\) and \(WF_P\)) onto the five operations \(A, K, H, T\) and \(I\) (on \(L\) and \(P\)), respectively.

If \(\pi_P(\xi) = 1\) in an interpretation \(\mathcal{I}\), \(\xi\) is said to be valid (or, true) in \(\mathcal{I}\), and we write \(\mathcal{I} \models \xi\). Let \(\Gamma\) be a theory over \(\text{AdjTPC}\), if \(\mathcal{I} \models \lambda\) for all \(\lambda \in \Gamma\), we write \(\mathcal{I} \models \Gamma\), and we say that \(\mathcal{I}\) is a model of \(\Gamma\). If for every interpretation \(\mathcal{I}\) such that \(\mathcal{I} \models \Gamma\) we have \(\mathcal{I} \models \delta\), then we write \(\Gamma \models \delta\). If \(\mathcal{I} \models \xi\) for all interpretations \(\mathcal{I}\) of \(\text{AdjTPC}\), we say that \(\xi\) is universally valid (or, a tautology), and we write \(\models \xi\).

**Theorem 2.4.1 (Soundness).** Let \(\Gamma\) be a theory over \(\text{AdjTPC}\) and \(\tau\) be a formula in \(WF_P\). If \(\Gamma \vdash \tau\), then \(\Gamma \vdash \tau\). Consequently, \(\text{AdjTPC}\) is sound for its semantics, in the sense that if \(\Gamma \vdash \tau\) then \(\models \tau\); that is, all theorems are universally valid.

**Proof.** By the identities N1–N7, we know that the axioms P1–P7 are universally valid in ADJT. Let \(\mathcal{A} = (A, \pi_L, \pi_P)\) be an interpretation such that \(\mathcal{A} \vdash \Gamma\). If a formula \(\xi\) has a deduction in one line from \(\Gamma\), then \(\xi\) is either an axiom or a member of \(\Gamma\). In both cases, \(\xi\) is valid in \(\mathcal{A}\). We now use complete induction on the number of lines in a derivation from \(\Gamma\). The induction hypothesis is that all formulae derived from \(\Gamma\) in \(n\) or fewer lines are valid in \(\mathcal{A}\). Suppose \(\tau\) has minimal derivation \(D\) from \(\Gamma\) in \(n + 1\) lines. By the minimality of \(D\), \(\tau\) is obtained in its last line by applying MP, then two of the previous lines in \(D\) must be \(\xi, \xi \rightarrow \tau\); for some formula \(\xi\). So by induction hypothesis, \(\pi_P(\xi) = 1\) and \(1 = \pi_P(\xi \rightarrow \tau) = I(\pi_P(\xi), \pi_P(\tau))\). It follows that \(1 = \pi_P(\xi) \leq p \pi_P(\tau)\); that is \(\pi_P(\tau) = 1\), which means \(\mathcal{A} \models \tau\). This establishes the induction step, and completes the proof. \(\square\)

We next address the question of the completeness of \(\text{AdjTPC}\) for ADJT. We follow a standard procedure due to Lindenbaum and Tarski.

**Lemma 2.4.1.** Let \(\Gamma\) be a theory over \(\text{AdjTPC}\) and let \(\xi\) be a formula in \(WF_P\) such that \(\Gamma \vdash \xi\). Then \(\Gamma \vdash \xi \rightarrow \xi\) for any formula \(\xi\) in \(WF_P\). In particular, \(\Gamma \vdash \xi \iff \xi\) will hold if and only if \(\Gamma \vdash \xi\) also holds.

**Proof.** From R3, \(\vdash \xi \rightarrow (\xi \rightarrow \xi)\). The lemma now follows by MP when \(\Gamma \vdash \xi\). \(\square\)

We construct the natural interpretation of \(\text{AdjTPC}\). Let \(\Gamma \subseteq WF_P\) be a fixed theory over \(\text{AdjTPC}\). The meta-predicate \(\Gamma \vdash \xi \iff \xi\) is an equivalence relation on \(WF_P\), denote it simply by \(\sim_p\) and call it \(\Gamma\)-equivalidity in \(WF_P\). Also, the meta-predicate \(\Gamma \vdash \beta \Implies \gamma\) is an equivalence relation on \(WF_L\), denote it simply by \(\sim_L\) and call it \(\Gamma\)-equiv-alidity in \(WF_L\). Let \(q : WF_L \rightarrow WF_L/ \sim_L : \alpha \mapsto [\alpha]^L\) and \(p : WF_P \rightarrow WF_P/ \sim_p : \xi \mapsto [\xi]^P\) be the quotient maps onto the two sets of all equivalence classes \([\alpha]^L = \{\beta \in WF_L : \Gamma \vdash \beta \Implies \alpha\}\) (formulae \(\Gamma\)-equivalent to \(\alpha\)) and \([\xi]^P = \{\tau \in WF_P : \Gamma \vdash \xi \Implies \tau\}\) (formulae \(\Gamma\)-equivalent to \(\xi\)), respectively.

The Substitution Theorem (Proposition 2.3.8) guarantees that the five logical connectives \(\Rightarrow, \&\), \(\lor\), \(\ast\), and \(\rightarrow\) possess the substitution property for the two \(\Gamma\)-equivaliditites \(\sim_L\) and \(\sim_p\). In consequence, the following binary operations \(\tilde{A}, \tilde{K}, \tilde{H}, \tilde{T},\) and \(\tilde{I}\) are well defined in \(WF_L/ \sim_L \cup WF_P/ \sim_p\): For all \([\beta]^L, [\gamma]^L\) in \(WF_L/ \sim_L\) and all \([\xi]^P, [\tau]^P\) in \(WF_P/ \sim_p\):

\[
\tilde{A}(\xi^P, \gamma^L) = q(\xi \Rightarrow \gamma),
\]
\[
\tilde{K}(\xi^P, [\gamma]^L) = q(\xi \& \gamma),
\]
\[
\tilde{H}(\beta^L, \gamma^L) = p(\beta \Implies \gamma),
\]
\[
\tilde{T}(\xi^P, \tau^P) = p(\xi \ast \tau),
\]
\[
\tilde{I}(\xi^P, \tau^P) = p(\xi \rightarrow \tau).
\]
Also, two partial orders \( \leq_L \) and \( \leq_P \) are well-defined on \( WF_L / \sim_L \) and \( WF_P / \sim_P \) by
\[
[x]_L^I \leq_L [\beta]_L^I \quad \text{iff} \quad \Gamma \vdash x \supset \beta, \quad (42)
\]
\[
[\xi]_P^P \leq_P [\tau]_P^P \quad \text{iff} \quad \Gamma \vdash \xi \rightarrow \tau. \quad (43)
\]

**Lemma 2.4.2.** The structure
\[
A^I = (WF_L / \sim_L, \leq_L, WF_P / \sim_P, \leq_P, \sim_1, A, \tilde{K}, \tilde{H}, \tilde{T}, \tilde{I}) \quad (44)
\]
is the quotient, above, of the tuple \((WF_L, \vdash \supset \subset \bullet, WF_P, \vdash \supset \rightarrow \bullet, \& \supset \cdot \cdot \& \rightarrow)\) with respect to the two relations of \( I^* \)-equivalidity in \( WF_P \) and \(WF_L\). It is a tied adjointness algebra, and the top element \( \tilde{1} \) of the poset \((WF_P / \sim_P, \leq_P)\) is the equivalence class \( \{\tau \in WF_P : \Gamma \vdash \tau\} \) of all theorems in \( \Gamma \). \( A^I \) together with the two quotient maps is a model of \textit{AdjTPC}. We call it the natural interpretation of \( \Gamma \) in \textit{AdjTPC}.

**Proof.** The assertion on the top element of \( WF_P / \sim_P \) follows from Lemma 2.4.1. Using Residuation, Proposition 2.3.3 and parts from Theorem 2.3.1, we find that \((WF_P / \sim_P, \leq_P, \tilde{1}, \tilde{T}, \tilde{I})\) is a residuated algebra.

Propositions 2.3.7, 2.3.6 ensure that \( \tilde{A} \) is an implication and \( \tilde{K} \) is a conjunction on \((WF_L / \sim_L, WF_P / \sim_P)\). Also, Adjointsness states, for all formulae \( \beta, \gamma \) and \( \xi \) that
\[
\Gamma \vdash \beta \supset (\xi \Rightarrow \gamma) \quad \text{iff} \quad \Gamma \vdash \xi \& \beta \supset \gamma \quad \text{iff} \quad \Gamma \vdash \xi \rightarrow (\beta \supset \gamma);
\]
that is,
\[
[\beta]_L^I \leq_L \tilde{A}([\xi]_P^P, [\gamma]_L^I) \quad \text{iff} \quad \tilde{K}([\xi]_P^P, [\beta]_L^I) \leq_L [\gamma]_L^I \quad \text{iff} \quad [\xi]_P^P \leq_P \tilde{H}([\beta]_L^I, [\gamma]_L^I).
\]
This means that \( \tilde{A}, \tilde{K}, \tilde{H} \) satisfy \textit{Adjointsness}. Moreover, (24) ensures that \( \tilde{T} \) ties the implication \( \tilde{A} \).

Finally, by their construction, \( \tilde{A}, \tilde{K}, \tilde{H}, \tilde{T}, \tilde{I}, p \) and \( q \) satisfy conditions (32)–(36) for \( p \) and \( q \) to become valuation functions. Thus, the tuple \((A^I, p, q)\) is a model of \( \Gamma \). \( \square \)

For any formula \( \xi \), we have \((A^I, p, q) \vdash \xi \) if and only if \( \Gamma \vdash \xi \). Therefore, in the light of the soundness theorem, we have

**Theorem 2.4.2 (Completeness).** Let \( \Gamma \) be a theory over \textit{AdjTPC}. Then for any formula \( \xi \) in \( WF_P \), \( \Gamma \vdash \xi \) if and only if \( \Gamma \vdash \xi \).

**Corollary 2.4.1.** (i) Two formulae \( \beta, \gamma \) in \( WF_L \) will be \( \Gamma \)-equivalid if and only if \( \pi_L(\beta) = \pi_L(\gamma) \) in all models of \( \Gamma \).
(ii) Two formulae \( \xi, \tau \) in \( WF_P \) will be \( \Gamma \)-equivalid if and only if \( \pi_P(\xi) = \pi_P(\tau) \) in all models of \( \Gamma \).

3. The prelinear tied propositional calculus

3.1. Syntax

In a tied adjointness lattice on \((L, P)\), the meet and join operations on \( P \) will be denoted by \( \wedge \) and \( \vee \), and on \( L \) by \( \tilde{\wedge} \) and \( \tilde{\vee} \).

**Definition 3.1.1 (Morsi et al. [15]).** A prelinear tied adjointness algebra is a tied adjointness lattice \( A = (L, \leq_L, P, \leq_P, 1, A, K, H, T, I, \wedge, \vee, \tilde{\wedge}, \tilde{\vee}) \) satisfying the following two prelinearity equations for \( I \) and \( H \):
\[
\forall a, c \in P : \quad I(a, c) \vee I(c, a) = 1, \quad (45)
\]
\[ \forall x, y \in L : H(x, y) \lor H(y, x) = 1. \quad (46) \]

We denote the class of all prelinear tied adjointness algebras by L-ADJT.

We aim to develop a complete syntax for the semantic domain L-ADJT. We call it Propositional Calculus for Prelinear Tied Adjointness Algebras, abbreviated to L-AdjTPC.

The language of L-AdjTPC is the language of AdjTPC, expanded by the connectives \( \land, \neg \). The connective \( \land : WF_P \times WF_P \rightarrow WF_P \), is called a weak conjunction on \( WF_P \). The connective \( \neg : WF_L \times WF_L \rightarrow WF_L \), is called a conjunction on \( WF_L \). Further definable connectives are

\[
\zeta \lor \tau = ((\zeta \rightarrow \tau) \rightarrow \tau) \land ((\tau \rightarrow \zeta) \rightarrow \zeta) \quad [4,9],
\]

\[
\alpha \lor \beta = ((\alpha \supset \beta) \rightarrow \beta) \neg ((\beta \supset \alpha) \rightarrow \alpha) \quad (\text{cf. [15]}).
\]

The connectives \( \lor \) and \( \lor \) are called disjunctions on \( WF_P \) and \( WF_L \), respectively.

We select the axioms for L-AdjTPC from among the many inequalities derived algebraically in L-ADJT. Since the logic L-AdjTPC is an extension of AdjTPC, the seven axioms of AdjTPC can be adopted, and we choose eight new axioms, namely, the following universally valid inequalities in L-ADJT:

From lattice properties we have for all \( a, b, c \) in \( P \) and \( x, y \) in \( L \)

**M8:** \( a \land b \leq p a \),

**M9:** \( a \land b \leq p b \land a \),

**M11:** \( x \land y \leq L x \),

**M12:** \( x \land y \leq L y \land x \). Also, from Theorems 2.3.3, 2.3.7 and the lattice properties, we have

**M10:** \( T(a, I(a, b)) \leq p a \land b \),

**M13:** \( K(H(x, y), x) \leq L x \land \neg y \).

From prelinearity we have \([15, \text{Lemma } 6.7]\)

**M14:** \( I(H(x, y), c) \leq p I(I(H(y, x), c), c) \),

**M15:** \( I(I(a, b), c) \leq p I(I(b, a), c), c) \).

Their corresponding statements on formulae, together with the seven axioms of AdjTPC, are what follows:

**Axioms of L-AdjTPC:**

\( P1: \vdash \xi \rightarrow (\tau \rightarrow \tau) \).

\( P2: \vdash (\xi \rightarrow (\tau \rightarrow \zeta)) \rightarrow ((\xi \land \tau) \rightarrow \zeta) \).

\( P3: \vdash (\xi \land \tau) \rightarrow (\eta) \rightarrow ((\eta \rightarrow \zeta) \rightarrow (\tau \rightarrow (\xi \rightarrow \zeta))). \)

\( P4: \vdash \xi \rightarrow (\beta \supset \beta) \).

\( P5: \vdash (\xi \rightarrow (\beta \supset \gamma)) \rightarrow ((\xi \land \beta) \supset \gamma) \).

\( P6: \vdash (\xi \land \beta) \supset (\xi \land \gamma) \rightarrow ((\xi \land \beta) \supset \gamma)). \)

\( P7: \vdash (\beta \supset (\xi \rightarrow \gamma)) \rightarrow (\xi \rightarrow (\beta \supset \gamma)) \).

\( P8: \vdash \xi \land \tau \rightarrow \xi \).

\( P9: \vdash \xi \land \tau \rightarrow \tau \land \xi \).

\( P10: \vdash (\xi \land \tau) \rightarrow (\tau \rightarrow \tau) \).

\( P11: \vdash (\xi \rightarrow \tau) \rightarrow (\xi \rightarrow \tau) \).

\( P12: \vdash (\xi \rightarrow \tau) \rightarrow (\xi \rightarrow \tau) \).

\( P13: \vdash (\xi \rightarrow \tau) \rightarrow (\xi \rightarrow \tau) \).

\( P14: \vdash (\xi \rightarrow (\tau \rightarrow \tau) \rightarrow (\xi \rightarrow (\tau \rightarrow \tau)) \rightarrow (\tau \rightarrow \tau) \rightarrow (\tau \rightarrow \tau)). \)

\( P15: \vdash (\xi \rightarrow (\tau \rightarrow \tau) \rightarrow (\xi \rightarrow (\tau \rightarrow \tau)) \rightarrow (\tau \rightarrow \tau) \rightarrow (\tau \rightarrow \tau)). \)

**Inference Rule for L-AdjTPC:**

**MP:** \( \xi, \xi \rightarrow \tau \rightarrow \tau \rightarrow (\xi, \tau \in WF_P) \) (Modus Ponens for R-implication).

We have taken axioms P8, P9, P10 and P15 from the axiom system for MTL due to Esteva and Godo \([4]\), and imitated them in axioms P11, P12, P13 and P14.

The theorems of AdjTPC remain valid in its enrichment L-AdjTPC. We derive further theorems in L-AdjTPC to establish its completeness for the semantic domain L-ADJT of prelinear tied adjointness algebras.
3.2. Theorems from MTL

It is direct to verify that our axioms on \((WF_P, \star, \rightarrow, \land, \lor)\), only, are equivalent through \(\mathbf{MP}\) to the axioms of the Logic of Basic Semi-hoops (HMTL) of Esteva et al. [5]. The Monoidal t-norm Based Logic (MTL), of Esteva and Godo [4], is the extension of HMTL by adding the truth constant \(\text{Falsum}\) and the axiom (A7: \(\vdash \text{Falsum} \rightarrow \tau\)) [5], both of which are absent from our system. This extension is conservative [5, Theorem 4.5]. In consequence, all theorems and deductions in MTL that do not deal with \(\text{Falsum}\) are perforce theorems and deductions in the residuated part \((WF_P, \star, \rightarrow, \land, \lor)\) of L-AdjTPC (and vice-versa), including what follows.

**Theorem 3.2.1** (Esteva and Godo [4], Hájek [8]). *MTL, and hence L-AdjTPC, prove the following properties of weak conjunction and disjunction on WF_P:* 

\[
\vdash \zeta \land \tau \rightarrow \zeta, \quad \vdash \zeta \land \tau \rightarrow \tau, \quad (49)
\]

\[
\vdash \zeta \rightarrow \zeta \lor \tau, \quad \vdash \tau \rightarrow \zeta \lor \tau, \quad (50)
\]

\[
\vdash \zeta \land (\zeta \lor \tau) \leftrightarrow \zeta, \quad \vdash \zeta \lor (\zeta \land \tau) \leftrightarrow \zeta, \quad (51)
\]

\[
\vdash \zeta \land \tau \rightarrow \tau \land \zeta, \quad \vdash \zeta \lor \tau \rightarrow \tau \lor \zeta, \quad (52)
\]

\[
\vdash \zeta \land \zeta \leftrightarrow \zeta, \quad \vdash \zeta \lor \zeta \leftrightarrow \zeta, \quad (53)
\]

\[
\vdash (\zeta \land \tau) \land \zeta \leftrightarrow \zeta \land (\tau \land \zeta), \quad (54)
\]

\[
\vdash (\zeta \lor \tau) \lor \zeta \leftrightarrow \zeta \lor (\tau \lor \zeta), \quad (55)
\]

\[
\vdash (\zeta \rightarrow \zeta) \land (\zeta \rightarrow \tau) \leftrightarrow (\zeta \rightarrow \zeta \land \tau), \quad (56)
\]

\[
\vdash (\zeta \rightarrow \tau) \leftrightarrow (\zeta \rightarrow \zeta \land \tau), \quad (57)
\]

\[
\vdash (\zeta \rightarrow \zeta) \land (\tau \rightarrow \zeta) \leftrightarrow (\zeta \lor \tau \rightarrow \zeta), \quad (58)
\]

\[
\vdash (\zeta \rightarrow \tau) \leftrightarrow (\zeta \lor \tau \rightarrow \tau), \quad (59)
\]

\[
\vdash (\zeta \rightarrow \zeta) \star (\zeta \rightarrow \tau) \rightarrow (\zeta \rightarrow \zeta \land \tau), \quad (60)
\]

\[
\vdash (\zeta \rightarrow \zeta) \star (\tau \rightarrow \zeta) \rightarrow (\zeta \lor \tau \rightarrow \zeta), \quad (61)
\]

\[
\vdash (\zeta \rightarrow \tau) \rightarrow (\zeta \lor \zeta \rightarrow \tau \lor \zeta), \quad (62)
\]

\[
\vdash (\zeta \rightarrow \tau) \rightarrow (\zeta \land \zeta \rightarrow \tau \land \zeta), \quad (63)
\]

\[
\vdash \star \tau \rightarrow \zeta \land \tau, \quad (64)
\]

\[
\vdash (\zeta \land \tau) \star (\zeta \land \tau) \rightarrow (\zeta \star \zeta) \land (\tau \star \tau), \quad (65)
\]

\[
\vdash \zeta \star (\tau \lor \zeta) \leftrightarrow (\zeta \star \tau) \lor (\zeta \star \zeta), \quad (66)
\]

\[
\vdash (\zeta \rightarrow \tau) \lor (\tau \rightarrow \zeta). \quad (67)
\]

**Proposition 3.2.1** (*Monotonicity, Esteva and Godo [4], Hájek [8]).

\[
\zeta \rightarrow \tau \vdash \{\zeta \lor \zeta \rightarrow \tau \lor \zeta, \zeta \lor \zeta \rightarrow \zeta \lor \tau, \zeta \land \zeta \rightarrow \tau \land \zeta, \zeta \land \zeta \rightarrow \zeta \land \tau\}. \quad (68)
\]

**Proposition 3.2.2** (*Esteva and Godo [4], Hájek [8]).

\[
\{\zeta \rightarrow \zeta, \zeta \rightarrow \tau\} \equiv \zeta \rightarrow \zeta \land \tau, \quad (69)
\]

\[
\{\zeta \rightarrow \zeta, \tau \rightarrow \zeta\} \equiv \zeta \lor \tau \rightarrow \zeta, \quad (70)
\]
\{ \xi, \tau \} \equiv \xi \land \tau \equiv \xi \star \tau. \quad (71)

\xi \vdash \xi \lor \tau, \quad (72)

\tau \vdash \xi \iff \xi \land \tau. \quad (73)

We use the abbreviation \( \tau^n \) to denote \( \tau \star \cdots \star \tau \) (n times).

**Theorem 3.2.2** *(Esteva and Godo [4], Hájek [8]).* 

\[ \vdash \xi \star \tau \rightarrow \xi^2 \lor \tau^2, \quad (74) \]

\[ \vdash (\xi \lor \tau) \star (\xi \lor \tau) \rightarrow \xi^2 \lor \tau^2. \quad (75) \]

**Proof (Hájek [8]).** By (63), we have \( (\xi \rightarrow \tau) \rightarrow (\xi \star \tau \rightarrow \tau^2) \), and so \( (\xi \rightarrow \tau) \rightarrow (\xi \star \tau \rightarrow \xi^2 \lor \tau^2) \). Similarly, \( (\tau \rightarrow \xi) \rightarrow (\xi \star \tau \rightarrow \xi^2 \lor \tau^2) \). Combine the last two results by axiom P15, to get (74).

Next, by (66), \( (\xi \lor \tau) \star (\xi \lor \tau) \iff \xi^2 \lor \tau^2 \lor \xi \star \tau \). Combine with (74) to obtain (75). \( \square \)

**Corollary 3.2.1** *(Esteva and Godo [4], Hájek [8]).* For each natural number \( n \): \( \xi \lor \tau \vdash \xi^n \lor \tau^n \).

3.3. **New theorems in L-AdjTPC**

We derive provabilities that deal also with the conjunction and disjunction on \( WFL \); putting into action also the axioms that govern them.

**Theorem 3.3.1** *(Commutativity).*

\[ \vdash \alpha \land \beta \subseteq \beta \land \alpha, \quad \vdash \alpha \lor \beta \subseteq \beta \lor \alpha. \quad (76) \]

**Proof.** These are obvious from P12 and from the definition of \( \lor \). \( \square \)

Prelinearity will enter into our proofs mainly through the two inferences of the next proposition.

**Proposition 3.3.1** *(Deduction by cases).*

\[ \{ (\alpha \supset \beta) \rightarrow \chi, (\beta \supset \alpha) \rightarrow \chi \} \vdash \chi, \quad (77) \]

\[ \{ (\xi \rightarrow \tau) \rightarrow \chi, (\tau \rightarrow \xi) \rightarrow \chi \} \vdash \chi. \quad (78) \]

**Proof.** These follow clearly from P14, P15 by MP. \( \square \)

**Theorem 3.3.2.**

\[ \vdash (\alpha \supset \beta) \iff (\alpha \supset (\alpha \supset \beta)), \quad (79) \]

\[ \vdash (\alpha \supset \beta) \land (\alpha \supset \gamma) \iff (\alpha \supset (\alpha \supset \beta \supset \gamma)), \quad (80) \]

\[ \vdash (\alpha \supset \beta) \star (\alpha \supset \gamma) \rightarrow (\alpha \supset (\beta \supset \gamma)). \quad (81) \]

**Proof.** By P13 and Adjointness, \( (\alpha \supset \beta) \rightarrow (\alpha \supset (\alpha \supset \beta)) \).

The other direction in (79) follows from P11 and the monotonicity of \( \supset \).

By (79) and R17

\( (\alpha \supset \beta) \star (\beta \supset \gamma) \rightarrow (\alpha \supset (\beta \supset \gamma)). \)

Hence by (10), \( (\alpha \supset \beta) \star (\beta \supset \gamma) \rightarrow (\alpha \supset (\beta \supset \gamma)), \)

and so by Residuation, \( (\beta \supset \gamma) \rightarrow ((\alpha \supset \beta) \rightarrow (\alpha \supset (\beta \supset \gamma))). \)

But by P8, \( (\alpha \supset \beta) \land (\alpha \supset \gamma) \rightarrow (\alpha \supset \beta). \) Hence, using R16,
Theorem 3.3.4. \((\beta \supset \gamma) \rightarrow ((x \supset \beta) \land (x \supset \gamma) \rightarrow (x \supset \beta \land \gamma))\).

By symmetry and \(P_9\), we also have
\((\gamma \supset \beta) \rightarrow ((x \supset \beta) \land (x \supset \gamma) \rightarrow (x \supset \beta \land \gamma))\)

Therefore, (77) yields \((x \supset \beta) \land (x \supset \gamma) \rightarrow (x \supset \beta \land \gamma)\).

Conversely, by \(P_{11}, P_9\) and the monotonicity of \(\supset\), \((x \supset \beta \land \gamma) \rightarrow (x \supset \beta) \land (x \supset \gamma)\). Hence, by applying Proposition 3.2.2 we obtain \((x \supset \beta \land \gamma) \rightarrow (x \supset \beta) \land (x \supset \gamma)\), and this completes the proof of (80); from which (81) follows by (64).

\[\Box\]

Proposition 3.3.2 (Greatest lower bound). \((\gamma \supset x, \gamma \supset \beta) \equiv \gamma \supset x \land \beta\).

Proof. By Proposition 3.2.2 and (81) of Theorem 3.3.2,

\[\{\gamma \supset x, \gamma \supset \beta\} \equiv (\gamma \supset x) \ast (\gamma \supset \beta) \vdash \gamma \supset x \land \beta.\]

The opposite inference follows from \(P_{11}\), the commutativity of \(\land\) and the monotonicity of \(\supset\).

\[\Box\]

Theorem 3.3.3.

\[\vdash x \supset x \land \beta,\]  \hspace{1cm} (82)

\[\vdash x \land \beta \supset (\beta \supset x) \Rightarrow x.\]  \hspace{1cm} (83)

Proof. By Theorem 2.3.7, \(x \supset ((x \supset \beta) \Rightarrow \beta)\), and by Theorem 2.3.7, \(x \supset ((\beta \supset x) \Rightarrow x)\). Hence by Proposition 3.3.2, \(x \supset ((x \supset \beta) \Rightarrow \beta)\) \(\land\) \(((\beta \supset x) \Rightarrow x)\) \(\Rightarrow x \supset \land \beta\).

Formula (83) follows from definition (48) of \(\land\) and \(P_{11}\).

\[\Box\]

The duality principle in \(\text{AdjTPC}\), adapted in Lemma 2.3.1 from Morsi [14], extends to \(\text{L-AdjTPC}\).

Lemma 3.3.1 (Duality). Let \(\Psi, \Phi\) be formulae in \(WF_P\) containing metavariables as atomic formulae. Let \(\Psi^d, \Phi^d\) be the formulae obtained from \(\Psi, \Phi\) by interchanging the two connectives \(\Rightarrow, \&\), interchanging the two connectives \(\land, \iff\), reversing the direction of the comparator \(\supset\) and keeping all other symbols fixed. Then in \(\text{L-AdjTPC}\), \(\Psi \vdash \Phi\) if and only if \(\Psi^d \vdash \Phi^d\). In particular, \(\Phi\) is a theorem in \(\text{L-AdjTPC}\) if and only if \(\Phi^d\) is a theorem in \(\text{L-AdjTPC}\).

Proof. This duality is self-inverse. Axioms \(P_1, P_2, P_3, P_4, P_8, P_9, P_{10}, P_{14}\) and \(P_{15}\) are self-dual, and we have shown in Lemma 2.3.1 that the duals of \(P_5, P_6, P_7\) are theorems in \(\text{AdjTPC}\). Moreover, the dual of \(P_{11}\) is (82), the dual of \(P_{12}\) is Theorem 3.3.1 and the dual of \(P_{13}\) is (83), and the inference rule \(\text{MP}\) is not affected by this duality. The assertion now follows by complete induction on the length of a derivation in \(\text{L-AdjTPC}\) of \(\Phi\) from \(\Psi\); the induction hypothesis being that the duals of its lines constitute a valid derivation of \(\Phi^d\) from \(\Psi^d\).

\[\Box\]

Theorem 3.3.4.

\[\vdash (x \supset \beta) \iff (x \iff \beta \supset \beta),\]  \hspace{1cm} (84)

\[\vdash (x \supset \gamma) \land (\beta \supset \gamma) \iff (x \iff \beta \supset \gamma),\]  \hspace{1cm} (85)

\[\vdash (x \supset \gamma) \ast (\beta \supset \gamma) \rightarrow (x \iff \beta \supset \gamma).\]  \hspace{1cm} (86)

Proof. By Duality from Theorem 3.3.2.

\[\Box\]

Proposition 3.3.3 (Least upper bound). \((x \supset \gamma, \beta \supset \gamma) \equiv x \iff \beta \supset \gamma\).

Proof. By Duality from Proposition 3.3.2.

\[\Box\]

Our axiom scheme is stable under the identification

\[WF_L = WF_P, \supset \Rightarrow \iff \land \& \ast,\]  \hspace{1cm} (87)
hence all our derivations are also stable under it. Accordingly, we shall refer to the process of deducing theorems in L-AdjTPC through identification (87) as **particularization to MTL**. □

**Theorem 3.3.5** *(Meet from join).*

\[ \vdash \alpha \sqcap \beta \sqsubseteq ((\beta \sqcup \alpha) \& \beta) \lor ((\alpha \sqcup \beta) \& \alpha), \]  
\[ (88) \]

\[ \vdash \xi \land \tau \iff (\tau \ast (\xi \rightarrow \eta)) \lor (\xi \ast (\tau \rightarrow \xi)). \]  
\[ (89) \]

**Proof.** Apply **Duality** to the definition of the connective \( \lor \). Then apply particularization to **MTL**. □

**Theorem 3.3.6.**

\[ \vdash (\alpha \sqcup \beta) \rightarrow (\alpha \sqcap \gamma \sqcup \beta \sqcap \gamma), \]  
\[ (90) \]

\[ \vdash (\alpha \sqcup \beta) \rightarrow (\alpha \sqcup \gamma \sqcup \beta \sqcup \gamma). \]  
\[ (91) \]

**Proof.** As \( \alpha \sqcap \gamma \sqcup \gamma \) (P11), then by (73),

\[ (\alpha \sqcap \gamma \sqcup \beta) \rightarrow (\alpha \sqcap \gamma \sqcup \beta) \land (\alpha \sqcap \gamma \sqcup \gamma). \]

Consequently, by (80), \( (\alpha \sqcap \gamma \sqcup \beta) \rightarrow (\alpha \sqcap \gamma \sqcup \beta \sqcap \gamma) \). But \( (\alpha \sqcup \beta) \rightarrow (\alpha \sqcap \gamma \sqcup \beta) \) by P11 and (9). Hence, (90) holds, and (91) follows by **Duality**. □

**Proposition 3.3.4** *(Monotonicity).*

\[ \alpha \sqcup \beta \vdash \{ \alpha \sqcap \gamma \sqcup \beta \sqcap \gamma, \gamma \sqcup \alpha \sqcup \gamma \sqcap \beta, \alpha \sqcup \gamma \sqcup \beta \sqcup \gamma, \gamma \sqcup \alpha \sqcup \gamma \sqcup \beta \}. \]  
\[ (92) \]

**Proof.** This follows from Theorem 3.3.6 by **MP**. □

From all the monotonicity propositions of the nine logical connectives \( \rightarrow, \& , \sqcup , \ast , \land , \lor \), and \( \sqcap \), we deduce the following Substitution Theorem in L-AdjTPC:

**Proposition 3.3.5** *(Substitution Theorem).* For all formulae \( \Psi_L \in WF_L \) and \( \Phi_p \in WF_P \):

\[ \xi \iff \tau, \alpha \sqsubseteq \beta \vdash \Psi_L (\alpha, \xi) \sqsubseteq \Psi_L (\alpha / \beta, \xi / \tau), \]

\[ (93) \]

**Proposition 3.3.6** *(Lattice order).*

\[ \alpha \sqcap \beta \sqsubseteq \alpha \equiv \alpha \lor \beta \sqsubseteq \beta \equiv \alpha \sqcup \beta. \]  
\[ (94) \]

**Proof.** These equivalences follow by P11, (79), (82) and (84). □

**Theorem 3.3.7.**

**Idempotence** : \( \vdash \alpha \sqcup \alpha \sqsubseteq \alpha \) and \( \vdash \beta \sqcap \beta \sqsubseteq \beta. \)  
\[ (95) \]

**Compatibility** : \( \vdash \alpha \sqcap (\alpha \lor \beta) \sqsubseteq \alpha \) and \( \vdash \alpha \lor (\alpha \sqcap \beta) \sqsubseteq \alpha. \)  
\[ (95) \]

**Prelinearity of** \( \lor \) : \( \vdash (\alpha \lor \beta) \lor (\beta \lor \alpha), \)

\[ \vdash (\alpha \lor \beta)^n \lor (\beta \lor \alpha)^n, \]  
\[ (96) \]

\[ \vdash (\xi \rightarrow \tau)^n \lor (\tau \rightarrow \xi)^n \]  
\[ (97) \]

for each natural number \( n \).
Proof. Equivalidities (94) follow from (93) together with Theorem 2.3.2. Equivalidities (95) are applications of (93), using (82) and P11. To prove (96), take $\chi = (x \supset y) \lor (y \supset x)$ in P14, and use (50) and MP. The last two theorems ensue by using (96) and (67) as premises in Corollary 3.2.1.  □

Proposition 3.3.7.

$$(a \lor b) \lor d \supseteq \forall \exists \{a \supseteq \exists, b \supseteq \forall, d \supseteq \forall\} \equiv a \lor (b \lor d) \supseteq \forall, \quad (99)$$

$$(\forall \exists \lor b) \lor d \supseteq \forall \exists \{\forall \exists \supseteq a, \exists \lor \forall, d \supseteq \forall\} \equiv \forall \exists (b \lor d) \supseteq \forall \exists. \quad (100)$$

Proof. These equivalences follow easily from Propositions 3.3.2 and 3.3.3.  □

Theorem 3.3.8 (Associativity).

$$\vdash (a \lor b) \lor d \supseteq a \lor (b \lor d), \quad (101)$$

$$\vdash (a \lor b) \lor d \supseteq a \lor (b \lor d) \lor c. \quad (102)$$

Proof. To derive (101), apply (99) twice, with $\frac{(a \lor b) \lor d}{\lor}$ and with $\frac{a \lor (b \lor d)}{\lor}$. Equivalidity (102) is proved similarly, using (100).  □

Theorem 3.3.9 (Lattice homomorphisms).

$$\vdash (\xi \Rightarrow \exists \beta \Rightarrow \gamma) \supseteq (\xi \Rightarrow \beta) \Rightarrow (\xi \Rightarrow \gamma), \quad (103)$$

$$\vdash (\xi \Rightarrow \beta \Rightarrow \gamma) \supseteq (\xi \Rightarrow \beta) \Rightarrow (\xi \Rightarrow \gamma), \quad (104)$$

$$\vdash (\xi \Rightarrow \beta \Rightarrow \gamma) \supseteq (\xi \Rightarrow \beta) \Rightarrow (\xi \Rightarrow \gamma), \quad (105)$$

$$\vdash (\xi \Rightarrow \beta \Rightarrow \gamma) \supseteq (\xi \Rightarrow \beta) \Rightarrow (\xi \Rightarrow \gamma), \quad (106)$$

$$\vdash (\xi \lor \beta \Rightarrow \gamma) \leftrightarrow (\xi \lor \beta) \lor (\xi \lor \gamma), \quad (107)$$

$$\vdash (\xi \lor \beta \Rightarrow \gamma) \leftrightarrow (\xi \lor \beta) \lor (\xi \lor \gamma), \quad (108)$$

$$\vdash (\xi \lor \beta \Rightarrow \gamma) \leftrightarrow (\xi \lor \beta) \lor (\xi \lor \gamma), \quad (109)$$

$$\vdash (\xi \lor \beta \Rightarrow \gamma) \leftrightarrow (\xi \lor \beta) \lor (\xi \lor \gamma), \quad (110)$$

$$\vdash \xi \land (\beta \lor \gamma) \subset \subset \xi \land \beta \lor (\xi \land \gamma), \quad (111)$$

$$\vdash (\xi \lor \tau \land \gamma) \subset \subset (\xi \lor \tau) \land (\xi \lor \gamma), \quad (112)$$

$$\vdash (\xi \lor \beta \land \gamma) \subset \subset (\xi \lor \beta) \land (\xi \lor \gamma), \quad (113)$$

$$\vdash (\xi \lor \tau \land \gamma) \subset \subset (\xi \lor \tau) \land (\xi \lor \gamma), \quad (114)$$

$$\vdash (\xi \lor \tau \land \gamma) \subset \subset (\xi \lor \tau) \land (\xi \lor \gamma), \quad (115)$$

$$\vdash (\xi \lor \tau \land \gamma) \subset \subset (\xi \lor \tau) \land (\xi \lor \gamma), \quad (116)$$

$$\vdash (\xi \lor \tau \land \gamma) \subset \subset (\xi \lor \tau) \land (\xi \lor \gamma), \quad (117)$$

Proof. All assertions in this theorem and the next one follow from the completeness of L-AdjT for L-ADJT (Theorem 3.4.1) coupled with the lattice-subdirect-product representation of prelinear tied adjointness algebras [15]. However, we believe their syntactical proofs should be presented to the interested reader, as we here do.

On one hand, $\vdash (\xi \Rightarrow \beta \lor \gamma) \supseteq (\xi \Rightarrow \beta) \lor (\xi \Rightarrow \gamma)$ follows from P11, the commutativity of $\lor$, (17) and Proposition 3.3.2.
On the other hand, we have by **P11** \((\xi \Rightarrow \beta) \wedge (\xi \Rightarrow \gamma) \supset (\xi \Rightarrow \beta)\).

Hence by *Adjointsness*, \(\xi \& ((\xi \Rightarrow \beta) \wedge (\xi \Rightarrow \gamma)) \supset \beta\).

By symmetry in \(\beta, \gamma\), we also have \(\xi \& ((\xi \Rightarrow \beta) \wedge (\xi \Rightarrow \gamma)) \supset \gamma\).

Proposition 3.3.2 now secures \(\vdash \xi \& ((\xi \Rightarrow \beta) \wedge (\xi \Rightarrow \gamma)) \supset \beta \wedge \gamma\).

This yields (103) through *Adjointsness*.

That \(\vdash (\xi \vee \tau \Rightarrow \gamma) \supset (\xi \Rightarrow \gamma) \wedge (\tau \Rightarrow \gamma)\) follows from (50), (18) and Proposition 3.3.2.

Conversely, we apply *Adjointsness* to the following instance of **P11**

\((\xi \Rightarrow \gamma) \wedge (\tau \Rightarrow \gamma) \supset (\xi \Rightarrow \gamma)\), and get \(\xi \rightarrow ((\xi \Rightarrow \gamma) \wedge (\tau \Rightarrow \gamma)) \supset \gamma)\).

By symmetry, \(\tau \rightarrow ((\xi \Rightarrow \gamma) \wedge (\tau \Rightarrow \gamma)) \supset \gamma)\).

Proposition 3.2.2 combines these two theorems in the one theorem \(\xi \vee \tau \rightarrow ((\xi \Rightarrow \gamma) \wedge (\tau \Rightarrow \gamma)) \supset \gamma)\), from which we get by *Adjointsness* again,

\((\xi \Rightarrow \gamma) \wedge (\tau \Rightarrow \gamma) \supset (\xi \vee \tau \Rightarrow \gamma)\).

This completes the proof of (104).

From (17) and (84), we know that

\(\vdash (\beta \supset \gamma) \rightarrow ((\beta \vee \gamma \supset \gamma) \wedge (\beta \Rightarrow \gamma))\).

We combine and use (82) and the monotonicity of \(\supset\), to obtain \(\vdash (\beta \supset \gamma) \rightarrow ((\xi \Rightarrow \beta \vee \gamma) \supset (\xi \Rightarrow \beta) \vee (\xi \Rightarrow \gamma))\).

By symmetry in \(\beta, \gamma\), \(\vdash (\gamma \Rightarrow \beta) \rightarrow ((\xi \Rightarrow \beta \vee \gamma) \supset (\xi \Rightarrow \beta) \vee (\xi \Rightarrow \gamma))\).

Consequently, inference (77) yields \(\vdash (\xi \Rightarrow \beta \vee \gamma) \supset (\xi \Rightarrow \beta) \vee (\xi \Rightarrow \gamma)\).

The other direction follows immediately from (17), (82) and Proposition 3.3.3. This proves (105).

Next, from (57) and (18), we know that

\(\vdash (\xi \rightarrow \tau) \rightarrow ((\xi \rightarrow \tau) \wedge (\xi \rightarrow \tau) \rightarrow ((\xi \wedge \tau \Rightarrow \gamma) \supset (\xi \Rightarrow \gamma))\).

We combine and use (82) and the monotonicity of \(\wedge\), to obtain \(\vdash (\xi \rightarrow \tau) \rightarrow (\xi \wedge \tau \Rightarrow \gamma) \supset (\xi \Rightarrow \tau) \vee (\tau \Rightarrow \gamma)\).

By symmetry in \(\xi, \tau\), \(\vdash (\xi \rightarrow \tau) \rightarrow (\xi \wedge \tau \Rightarrow \gamma) \supset (\xi \Rightarrow \tau) \vee (\tau \Rightarrow \gamma)\).

Consequently, inference (78) yields \(\vdash (\xi \wedge \tau \Rightarrow \gamma) \supset (\xi \Rightarrow \gamma) \vee (\tau \Rightarrow \gamma)\).

The other direction follows immediately from **P11**, (18) and Proposition 3.3.3. This proves (106).

Equivalences (107) and (108) have been proved above.

Next, from (8) and (84), we know that

\(\vdash (\beta \supset \gamma) \rightarrow ((\beta \supset \gamma \supset \gamma) \wedge (\beta \supset \gamma))\).

We combine and use (82) and the monotonicity of \(\supset\), to obtain \(\vdash (\beta \supset \gamma) \rightarrow ((\alpha \supset \beta \supset \gamma) \rightarrow (\alpha \supset \beta))\).

By symmetry in \(\beta, \gamma\), \(\vdash (\gamma \supset \beta) \rightarrow ((\alpha \supset \beta \supset \gamma) \rightarrow (\alpha \supset \beta) \vee (\alpha \supset \gamma))\).

Consequently, inference (77) yields \(\vdash (\alpha \supset \beta \supset \gamma) \rightarrow (\alpha \supset \beta) \vee (\alpha \supset \gamma)\).

The other direction follows immediately from (8), (82) and Proposition 3.3.3. This proves (109).

Equivalences (110), (111), (112), (113) and (114) follow by *Duality*. Finally, (115), (116), and (117) hold by particularization to MTL.

See also the remaining lattice-homomorphism properties of \(\rightarrow\) and \(*\) in Theorem 3.2.1.

**Theorem 3.3.10 (Distributivity).**

\[\vdash \alpha \vee (\beta \wedge \delta) \supset (\alpha \vee \beta) \wedge (\alpha \vee \delta), \tag{118}\]

\[\vdash \alpha \wedge (\beta \vee \delta) \supset (\alpha \wedge \beta) \vee (\alpha \wedge \delta), \tag{119}\]

\[\vdash \xi \vee (\tau \wedge \xi) \leftrightarrow ((\xi \vee \tau) \wedge (\xi \vee \xi)) \tag{4, 9}, \tag{120}\]

\[\vdash \xi \wedge (\tau \vee \xi) \leftrightarrow ((\xi \wedge \tau) \vee (\xi \wedge \xi)) \tag{4, 9}. \tag{121}\]

**Proof.** By combining (96) with (79), \(\vdash (\beta \supset \beta \wedge \delta) \vee (\delta \supset \beta \wedge \delta)\). Using this, monotonicity of \(\vee\) and \(\wedge\) (Proposition 3.2.1, 3.3.4) and MP we obtain \(\vdash (\alpha \vee \beta \supset \alpha \vee (\beta \wedge \delta)) \vee (\alpha \vee \delta \supset \alpha \vee (\beta \wedge \delta))\). Consequently, by substitution from (110), \(\vdash (\alpha \vee \beta) \wedge (\alpha \vee \delta) \supset \alpha \vee (\beta \wedge \delta)\). The other half of (118) follows from the monotonicity of \(\vee\).

Formula (119) follows by *Duality*. Then (120) and (121) follow by particularization to MTL. □
3.4. Completeness

Completeness of a logical system corresponding to an algebraic variety can be considered in (no less than) four different meanings [2,3,8,10]:

1. Completeness with respect to the whole variety (general completeness),
2. Completeness with respect to the class of chains of the variety (chain completeness),
3. Completeness with respect to the set of \([0, 1]\)-structures in the variety (standard completeness),
4. Completeness with respect to the set of structures in the variety over the rationals in \([0, 1]\) (rational completeness).

The last two notions are out of the scope of this work. We establish both the general and chain completeness of \(L-\text{AdjTPC}\) for the variety \(L-\text{ADJT}\) of prelinear tied adjointness algebras. Its soundness is clearly deduced from the arguments of Section 3.1.

**Definition 3.4.1.** For each theory \(\Gamma \subseteq WFP\) over \(L-\text{AdjTPC}\) we let

\[
A_\Gamma = (WFL, \sim_L, \leq_L, WFP, \sim_P, \leq_P, \tilde{I}, \tilde{\Lambda}, \tilde{\check{K}}, \tilde{\check{H}}, \tilde{\check{\check{r}}}, \check{\check{V}}, \tilde{T}, \tilde{I}, \tilde{\Lambda}, \tilde{\check{V}})
\]

(122)

be the algebra of classes of \(\Gamma\)-equivalid formulae; that is, the quotient of the tuple

\((WFL, \vdash \supset, WFP, \vdash \rightarrow, \{\Gamma\text{-theorems}\}, \Rightarrow, \& \supset, \check{\check{r}}, \check{\check{V}}, \check{\check{I}}, \check{\check{\check{I}}, \tilde{\Lambda}, \tilde{\check{V}}})\),

with respect to the two equivalence relations of \(\Gamma\)-equivalidity in \(WFL\) and in \(WFP\).

This \(A_\Gamma\) is well defined, due to the Substitution Theorem. It is a tied adjointness algebra (see Lemma 2.4.2), and its two partial orders (described in (42) and (43)) are its lattice orders, by virtue of Propositions 3.2.2, 3.3.6. It is prelinear because of (67) and (96), and the top element \(\tilde{T}\) of \(WFP, \sim_P\) is exactly the set of all \(\Gamma\)-provable formulae in \(L-\text{AdjTPC}\), by Lemma 2.4.1. Furthermore, the two quotient maps onto \(WFL, \sim_L\) and \(WFP, \sim_P\) are valuation functions.

From these arguments together with the representation theorem in Part 1 [15], we deduce

**Theorem 3.4.1 (Completeness).** \(L-\text{AdjTPC}\) is strongly generally complete and strongly chain complete for \(L-\text{ADJT}\). Specifically, let \(\Gamma\) be a theory over \(L-\text{AdjTPC}\) and \(\tau\) be a formula in \(WFP\). Then the following are equivalent:

(i) \(\Gamma \vdash \tau\).
(ii) For each \(L-\text{ADJT}\)-model \(\mathcal{A} = (\pi_L, \pi_P)\) of \(\Gamma\), \(\mathcal{A} \models \tau\).
(iii) For each linearly ordered \(L-\text{ADJT}\)-model \(\mathcal{A} = (\pi_L, \pi_P)\) of \(\Gamma\), \(\mathcal{A} \models \tau\).

We remark that in the logic \(BL\), Hájek [8] says that a theory \(\Gamma\) over \(BL\) is complete if for each pair \(\xi, \tau\) of formulae, \(\Gamma \vdash (\xi \rightarrow \tau)\) or \(\Gamma \vdash (\tau \rightarrow \xi)\). He then proves that every theory over \(BL\) has a complete supertheory, and proceeds from there to deduce his strong completeness theorem. The authors possess full details of an analogous approach within \(L-\text{AdjTPC}\), culminating in an alternative proof of Theorem 3.4.1.

4. The prelinear tied predicate logic

4.1. Languages and axioms

We are now ready to start our investigation of the predicate logic (or first order logic, quantification logic) \(L-\text{AdjT}\) of prelinear tied adjointness algebras, whereby atomic formulae have structure; each atom consists of a predicate and some terms forming its arguments. Object variables are particular terms; variables may be quantified using a universal quantifier \(\forall\) “for all”, and an existential quantifier \(\exists\) “there is”. In interpretations, predicates are interpreted by fuzzy relations (either \(P\)-valued or \(L\)-valued), and the quantifiers \(\forall\) and \(\exists\) are interpreted by inf and sup, respectively, in the disjoint union \(P \cup L\) of two lattices \(P\) and \(L\) of truth values.
Following binary predicate logic (see [17]) and BL-logic (Hájek [8, Chapter 5]), we begin our introduction of L-AdjTV by describing its languages.

**Definition 4.1.1.** A predicate language \( \mathcal{L} \) for L-AdjTV consists of two non-empty sets of predicates (each together with a positive natural number; its arity) \( \text{Pred}_p = \{ R_1, R_2, \ldots \} \) and \( \text{Pred}_Q = \{ Q_1, Q_2, \ldots \} \), a set of object variables \( \text{Var} = \{ x, y, \ldots \} \) and a (possibly empty) set of object constants \( \text{Const} = \{ c_1, c_2, \ldots \} \). Terms are object variables and object constants. We have two sets of atomic formulae. The set \( WF_{P_0} \) of atomic formulae that have the form \( R(t_1, \ldots, t_n) \), where \( R \in \text{Pred}_p \) is a predicate of some arity \( n \) and \( t_1, \ldots, t_n \) are terms, and the set \( WF_{L_0} \) of atomic formulae that have the form \( Q(t_1, \ldots, t_n) \), where \( Q \in \text{Pred}_Q \) is a predicate of some arity \( n \) and \( t_1, \ldots, t_n \) are terms. Well-formed formulae are obtained from \( WF_{P_0} \cup WF_{L_0} \) by repeated application of the connectives \( \rightarrow, \&., \lor, \land \), and \( \neg \), and the quantifiers \( \forall, \exists \), taking into account that now, if \( x \) is a formula in \( WF_L \) and \( x \) is an object variable, then \( (\forall x)x \) and \( (\exists x)x \) are formulae in \( WF_L \), also if \( \tau \) is a formula in \( WF_P \) and \( x \) is an object variable, then \( (\forall x)\tau \) and \( (\exists x)\tau \) are formulae in \( WF_P \). Logical symbols are the object variables, the connectives \( \Rightarrow, \&., \lor, \land \) and \( \neg \), and the quantifiers \( \forall, \exists \), together with the other connectives \( \lor, \land \) and symbols \( \leftrightarrow, \subset \) defined as above.

The notions of free/bound variables, of subformulae and of substitution of a term for a variable are understood as in the classical predicate logic, and can be looked up in [17] or in [8, Chapter 5].

**Axioms of L-AdjTV:** The axioms of L-AdjTV are schemata of formulae whose metavariables stand for arbitrary formulae of any predicate language for L-AdjTV. These include the schemata resulting in this way from the 15 axioms of L-AdjTPC, plus the following nine axioms on quantifiers (\( \xi, \tau \in WF_P, x, y \in WF_L \)):

\[
\begin{align*}
(\forall 1) & \ (\forall x)\xi(x) \rightarrow \xi(t) \quad (t \text{ substitutable for } x \text{ in } \xi(x)), \\
(\exists 1) & \ (\exists t)\xi(x) \rightarrow (\exists x)\xi(x) \quad (t \text{ substitutable for } x \text{ in } \xi(x)), \\
(\forall 2) & \ (\forall x)(\xi \rightarrow \tau) \rightarrow (\xi \rightarrow (\forall x)\tau) \quad (x \text{ not free in } \xi), \\
(\exists 2) & \ (\exists x)(\tau \rightarrow \xi) \rightarrow ((\exists x)\tau \rightarrow \xi) \quad (x \text{ not free in } \xi), \\
(\forall 3) & \ (\forall x)(\xi \lor \xi) \rightarrow ((\forall x)\xi \lor \xi) \quad (x \text{ not free in } \xi), \\
(\exists 3) & \ (\exists x)(\xi \lor x) \rightarrow ((\exists x)\xi \lor x) \quad (x \text{ not free in } \xi),
\end{align*}
\]

From now on, a theory \( \Gamma \) over L-AdjTV will consist of a language \( \mathcal{L} \) for L-AdjTV together with a set of formulae (in the set \( WF_P \) of \( \mathcal{L} \)) called the special axioms of that theory. An inference in \( \Gamma \), (a \( \Gamma \)-deduction, \( \Gamma \)-proof or \( \Gamma \)-derivation) \( \Gamma \cup \Theta \vdash \tau \) (where \( \Theta \) is some subset of \( WF_P \)) is understood in the usual way, by starting from the axioms of L-AdjTV and the special axioms of \( \Gamma \), as theorems in \( \Gamma \), and from the formulae in \( \Theta \), and by repeated application of the following two deduction rules:

**Inference Rules for L-AdjTV:**

- **Modus Ponens:** \( \Gamma \cup \{ \xi, \xi \rightarrow \tau \} \vdash \tau, \)
- **Generalization:** \( \Gamma \cup \{ \xi \} \vdash (\forall x)\xi, (\xi, \tau \in WF_P). \)

When \( \Gamma \vdash \tau \), we say that \( \tau \) is a theorem in \( \Gamma \) (a \( \Gamma \)-provability). When \( \xi \) is a formula scheme derived from the axioms of L-AdjTV only, by means of the two inference rules, we say that \( \xi \) is a theorem of L-AdjTV, and write \( \vdash \xi \). Such \( \xi \) is a theorem in every theory over L-AdjTV.

**4.2. Theorems in L-AdjTV**

**Lemma 4.2.1.** Duality extends to L-AdjTV, by further involving the interchange of the two parts of the quantifiers \( \forall \) and \( \exists \) on \( WF_L \) alone, whereas the other two parts \( \forall, \exists : WF_P \rightarrow WF_P \) are kept unchanged.

**Proof.** Combine the proof of Lemma 3.3.1 with the following: the dual of axiom (\( \forall 4 \)) is (\( \exists 3 \)), and the dual of axiom (\( \forall 5 \)) is (\( \exists 4 \)), whereas the other axioms on quantifiers and the two inference rules are not affected by this duality. □

The axioms (\( \forall 1 \)), (\( \forall 2 \)), (\( \forall 3 \)), (\( \exists 1 \)) and (\( \exists 2 \)) are the axioms on quantifiers of HMTLV [5], MTLv [4] and BLv [8]. Also, the only inference rule on quantifiers in both L-AdjTV and HMTLV is the same rule Generalization. By virtue
of this and our arguments at the beginning of Section 3.2, the residuated part of \( \mathbf{L-AdjTV} \) coincides with \( \mathbf{HMTL}_\Psi \).

Accordingly, as \( \mathbf{MTL}_\Psi \) is a conservative extension of \( \mathbf{HMTL}_\Psi \) [5, Theorem 5.5], the following formulae provable in \( \mathbf{MTL}_\Psi [4] \) remain theorems of \( \mathbf{L-AdjTV} \).

**Theorem 4.2.1** (Esteva and Godo [4]). Let \( \zeta \) and \( \tau \) be arbitrary formulae in \( \mathbf{WF}_\mathbb{P} \), and let \( \xi \) be a formula in \( \mathbf{WF}_\mathbb{P} \) such that \( x \) is not free in \( \xi \). Then \( \mathbf{L-AdjTV} \) proves the following:

\[
\vdash \xi \leftrightarrow (\forall x)\xi,
\]
\[
\vdash \xi \leftrightarrow (\exists x)\xi,
\]
\[
\vdash (\forall x)(\xi \rightarrow \tau) \leftrightarrow ((\forall x)\tau \rightarrow \xi),
\]
\[
\vdash (\forall x)(\tau \rightarrow \xi) \leftrightarrow ((\exists x)\tau \rightarrow \xi),
\]
\[
\vdash (\exists x)(\xi \rightarrow \tau) \rightarrow ((\forall x)\xi \rightarrow (\exists x)\tau),
\]
\[
\vdash (\exists x)(\tau \rightarrow \xi) \rightarrow ((\forall x)\tau \rightarrow (\exists x)\xi),
\]
\[
\vdash ((\forall x)(\xi \ast (\exists x)(\tau)) \rightarrow ((\exists x)(\xi \ast \tau)),
\]
\[
\vdash (\forall x)(\tau(x) \rightarrow (\forall y)\tau(y)), \text{ if } y \text{ is substitutable for } x \text{ in } \tau(x),
\]
\[
\vdash (\exists x)(\tau(x) \rightarrow (\exists y)\tau(y)), \text{ if } y \text{ is substitutable for } x \text{ in } \tau(x),
\]
\[
\vdash (\exists x)(\tau \ast \xi) \leftrightarrow ((\exists x)\tau \ast \xi),
\]
\[
\vdash (\exists x)(\tau \ast \xi) \leftrightarrow ((\exists x)\tau \ast (\exists x)\xi),
\]
\[
\vdash (\forall x)(\zeta \land \tau) \leftrightarrow ((\forall x)\zeta \land (\forall x)\tau),
\]
\[
\vdash (\forall x)(\zeta \lor \tau) \leftrightarrow ((\exists x)\zeta \lor (\exists x)\tau).
\]

Now we introduce new theorems and inferences that also need the other axioms (\( \forall 4 \)), (\( \forall 5 \)), (\( \exists 3 \)) and (\( \exists 4 \)) in their proofs.

**Theorem 4.2.2.** For arbitrary formulae \( \alpha \) and \( \gamma \) in \( \mathbf{WF}_\mathbb{L} \),

\[
\vdash (\forall x)(\alpha \supset \gamma) \rightarrow ((\forall x)\alpha \supset (\forall x)\gamma), \tag{123}
\]
\[
\vdash (\exists x)(\alpha \supset \gamma) \rightarrow ((\exists x)\alpha \supset (\exists x)\gamma). \tag{124}
\]

**Proof.** By (\( \forall 4 \)) and Proposition 2.3.7 we get

\[
\vdash (\alpha \supset \gamma) \rightarrow ((\forall x)\alpha \supset \gamma). \text{ Then by } (\forall 1), \vdash (\forall x)(\alpha \supset \gamma) \rightarrow ((\forall x)\alpha \supset \gamma).
\]

Generalize:

\[
\vdash (\forall x)((\forall x)(\alpha \supset \gamma) \rightarrow ((\forall x)\alpha \supset \gamma)). \text{ Hence by applying } (\forall 2) \text{ and then } (\forall 5) \text{ we get (123), from which we get (124) by Duality.} \]

**Proposition 4.2.1 (Monotonicity of quantifiers).**

\[
\alpha \supset \gamma \vdash ((\forall x)\alpha \supset (\forall x)\gamma), (\exists x)\alpha \supset (\exists x)\gamma, \tag{125}
\]
\[
\zeta \rightarrow \zeta \vdash ((\forall x)\zeta \rightarrow (\forall x)\zeta), (\exists x)\zeta \rightarrow (\exists x)\zeta) [4]. \tag{126}
\]

**Proof.** These follow from Theorems 4.2.2 and 4.2.1 by generalization and \( \mathbf{MP} \). \( \square \)

From this proposition, we conclude that the Substitution Theorem (cf. Proposition 3.3.5) holds in \( \mathbf{L-AdjTV} \).

In the proofs of the next four theorems, we use the monotonicity inferences in Proposition 2.3.7 repeatedly.

**Theorem 4.2.3.** Let \( \alpha \) be an arbitrary formula in \( \mathbf{WF}_\mathbb{L} \) and \( \gamma \) a formula in \( \mathbf{WF}_\mathbb{L} \) not containing \( x \) freely. Then

(i) \( \vdash \gamma \supset (\forall x)\gamma \),

(ii) \( \vdash \gamma \supset (\exists x)\gamma \),

(iii) \( \vdash (\forall x)(\gamma \supset \alpha) \leftrightarrow (\gamma \supset (\forall x)\alpha) \),

(iv) \( \vdash (\forall x)(\alpha \supset \gamma) \leftrightarrow ((\exists x)\alpha \supset \gamma) \).
Theorem 4.2.4. For arbitrary formulae \( \gamma \) we get \( \vdash (\forall x)(\gamma \supset (\exists x)\gamma) \).

Theorem 4.2.5. Let \( \gamma \) be an arbitrary formula in \( WF_L \) and \( \zeta \) a formula in \( WF_P \) not containing \( x \) freely. Then

(i) \( \vdash (\forall x)(\zeta \Rightarrow \gamma) \supset ((\forall x)\zeta \Rightarrow (\forall x)\gamma) \),
(ii) \( \vdash (\exists x)(\zeta \Rightarrow \gamma) \supset ((\exists x)\zeta \Rightarrow (\exists x)\gamma) \),
(iii) \( \vdash (\forall x)(\zeta & (\exists x)\gamma \supset ((\forall x)(\zeta & (\exists x)\gamma) \),
(iv) \( \vdash (\exists x)(\zeta & (\forall x)\gamma \supset ((\exists x)(\zeta & (\forall x)\gamma) \).

Proof. (i) By \( \forall 1 \) and \( \forall 4 \) we deduce
\[ \vdash (\forall x)(\zeta \Rightarrow \gamma) \supset ((\forall x)\zeta \Rightarrow (\forall x)\gamma) \],
and so \( \vdash (\forall x)(\zeta \Rightarrow \gamma) \supset (\forall x)\gamma \). Generalize and apply \( \forall 5 \) to get
\[ \vdash (\forall x)(\zeta \Rightarrow \gamma) \supset (\forall x)(\zeta \Rightarrow (\forall x)\gamma) \),
from which we get the result by \( Adjointness \).

(ii) Analogously, we have by \( \forall 4 \) and \( \exists 3 \),
\[ \vdash ((\exists x)\zeta \Rightarrow \gamma) \supset ((\exists x)(\zeta \Rightarrow \gamma) \supset (\exists x)\gamma \). Consequently, \( \vdash (\forall x)(\zeta \Rightarrow \gamma) \supset ((\forall x)(\zeta \Rightarrow \gamma) \supset (\exists x)\gamma) \). Generalize and apply \( \exists 2 \) to get
\[ \vdash (\exists x)(\zeta \Rightarrow ((\forall x)(\zeta \Rightarrow \gamma) \supset (\exists x)\gamma) \). The assertion now follows by \( Adjointness \).

(iii) and (iv) By \( Duality \). □

Theorem 4.2.6. Let \( \zeta \) be an arbitrary formula in \( WF_P \) and \( \gamma \) a formula in \( WF_L \) not containing \( x \) freely. Then

(i) \( \vdash (\forall x)(\zeta \Rightarrow \gamma) \supset ((\forall x)\zeta \Rightarrow (\forall x)\gamma) \),
(ii) \( \vdash (\exists x)(\zeta \Rightarrow \gamma) \supset ((\exists x)\zeta \Rightarrow (\exists x)\gamma) \),
(iii) \( \vdash (\forall x)(\zeta & (\exists x)\gamma \supset ((\forall x)(\zeta & (\exists x)\gamma) \),
(iv) \( \vdash (\exists x)(\zeta & (\forall x)\gamma \supset ((\exists x)(\zeta & (\forall x)\gamma) \).

Proof. (i) By \( \forall 4 \), \( \vdash (\zeta \Rightarrow (\forall x)\gamma) \supset (\zeta \Rightarrow (\exists x)\gamma) \). Generalize then apply \( \forall 5 \) to get
\[ \vdash (\zeta \Rightarrow (\forall x)\gamma) \supset (\forall x)(\zeta \Rightarrow (\forall x)\gamma) \).

The other direction follows from Theorem 4.2.4(i).

(ii) By \( \exists 3 \), \( \vdash (\zeta \Rightarrow (\exists x)\gamma) \supset (\zeta \Rightarrow (\exists x)\gamma) \). Generalize then use \( \exists 4 \) to get the result.

(iii) and (iv) By \( Duality \). □
Theorem 4.2.7. If \( y \) is substitutable for \( x \) in a formula \( \alpha(x) \) in \( WF_L \), then
\[
\vdash (\forall x)\alpha(x) \sqsupset (\forall y)\alpha(y),
\]
\[
\vdash (\exists x)\alpha(x) \sqsupset (\exists y)\alpha(y).
\]

Proof. Use generalization and \((\forall\gamma)\). \(\square\)

Theorem 4.2.8. Let \( \alpha \) and \( \gamma \) be arbitrary formulae in \( WF_L \) Then
\[
\vdash (\exists x)(\beta \sqsubseteq \gamma) \sqsupset (\exists x)\beta \sqsubseteq (\exists x)\gamma),
\]
\[
\vdash (\forall x)(\beta \sqsupset \gamma) \sqsupset (\forall x)\beta \sqsupset (\forall x)\gamma).
\]

Proof. By Proposition 4.2.1 then Proposition 3.3.3 we have \( \vdash ((\exists x)\beta \sqsubseteq (\exists x)\gamma) \supset (\exists x)(\beta \sqsubseteq \gamma). \) Conversely, Proposition 4.2.1 together with \((\exists x)\beta \sqsubseteq (\exists x)\gamma) \supset (\exists x)\beta \sqsubseteq (\exists x)\gamma). \) This proves (129). Equivalidity (130) then holds by Duality. \(\square\)

We now employ the axiom \((\forall\gamma)\). In the next theorem, we use a process of particularization to \( MTL\). This process in \( L-Adj(T) \) has the same meaning as that of particularization to \( MTL \) in \( L-Adj(T) \) (see the identification (87) in Section 3.3), with the added procedure that the quantifiers \( \forall, \exists \) on \( WF_P \) are identified with the quantifiers \( \forall, \exists \) on \( WF_L \), respectively.

Theorem 4.2.9. Let \( \alpha, \gamma \in WF_L \) and \( \tau, \zeta \in WF_P \) be such that neither \( \gamma \) nor \( \tau \) contains \( x \) freely. Then
\[
\vdash (\exists x)(\gamma \setminus x) \sqsupset (\exists x)\alpha,
\]
\[
\vdash (\forall x)(\gamma \setminus x) \sqsupset (\forall x)\alpha,
\]
\[
\vdash (\exists x)(\tau \wedge \zeta) \rightarrow (\exists x)\zeta [4, 8],
\]
\[
\vdash (\forall x)(\tau \vee \zeta) \rightarrow (\forall x)\zeta [4, 8].
\]

Proof. By \((\exists x)\) followed by generalization, \((\forall x)(\gamma \setminus x) \supset (\exists x)(\gamma \setminus x)). \) Hence by (110) \((\forall x)((\gamma \supset (\exists x)\gamma \wedge x)) \lor (x \supset (\exists x)(\gamma \wedge x))). \) Now we apply \((\forall\gamma)\) and obtain \((\gamma \supset (\exists x)(\gamma \wedge x)) \supset (\forall x)(x \supset (\exists x)(\gamma \wedge x)), \) from which \((\gamma \supset (\exists x)(\gamma \wedge x)) \lor ((\exists x)\alpha \supset (\exists x)(\gamma \wedge x)) \) follows by Theorem 4.2.3(iv). Finally by (110) again, \( \gamma \wedge (\exists x)\alpha \supset (\exists x)(\gamma \wedge x). \) The other direction ensues from the monotonicity of \( \exists \) (Proposition 4.2.1) followed by Proposition 3.3.2. This proves (131).

Equivalidity (132) then holds by Duality, and (133), (134) by particularization to \( MTL\). \(\square\)

Equivalities (133) and (134) are also provable in both \( BL\) [8] and \( MTL\) [4]. However, from among the proofs of (133), axiom \((\forall\gamma)\) has been indispensable in our proof only.

4.3. Completeness

We call a lattice \( P \) a semi-Heyting algebra if its meet distributes over existing arbitrary suprema; that is, for any \( a \in P \) and any subset \( \{b_i\}_{i \in \Omega} \subseteq P \):
\[
a \wedge \sup_{i \in \Omega} b_i = \sup_{i \in \Omega} (a \wedge b_i)
\]
whenever \( \{b_i\}_{i \in \Omega} \) has supremum in \( P \). (All Heyting algebras are semi-Heyting algebras, and the two notions coincide for complete lattices, cf. [6].) It is called a co-semi-Heyting algebra if it satisfies:
\[
a \vee \inf_{i \in \Omega} b_i = \inf_{i \in \Omega} (a \vee b_i)
\]
Whenever \( \{b_i\}_{i \in I} \) has infimum in \( P \), a lattice is called a bi-semi-Heyting algebra if it is simultaneously a semi-Heyting and co-semi-Heyting algebra. All chains are bi-semi-Heyting algebras. Eqs. (135) and (136) are not equivalent to each other [6].

**Definition 4.3.1.** An sH-prelinear tied adjointness algebra is a prelinear tied adjointness algebra over \((L, P)\) in which \( P \) is a co-semi-Heyting algebra. The class of all sH-prelinear tied adjointness algebras is denoted by \( sHL-\text{ADJT} \).

The proof of the next lemma traces the lines of that of Theorem 4.2.9. We supply it in order to show that the possible lack of completeness of the underlying lattices does not obstruct the conclusion.

**Lemma 4.3.1.** Both lattices in an sH-prelinear tied adjointness algebra are bi-semi-Heyting algebras.

**Proof.** Let \( A = (L, \leq_L, P, \leq_p, 1, A, K, H, T, I, \land, \lor, \neg, \top) \) be an sH-prelinear tied adjointness algebra. Let \( \{z_j\}_{j \in I} \) be a nonempty subset of \( L \) that has a supremum in \( L \), and let \( y \in L \). Then for any upper bound \( w \) of the set \( \{y \land z_j\}_{j \in I} \), we have by (110), (3), and because \( P \) is a co-semi-Heyting algebra (by definition),

\[
H\left( y \land \sup_{j \in I} z_j, w \right) = H(y, w) \lor H\left( \sup_{j \in I} z_j, w \right) \\
= H(y, w) \lor \inf_{j \in I} H(z_j, w) \\
= \inf_{j \in I} (H(y, w) \lor H(z_j, w)) \\
= \inf_{j \in I} H(y \land z_j, w) \text{ (by (110) again)} \\
= 1 \text{ by (1)}.
\]

So by (1), \( y \land \sup_{j \in I} z_j \leq w \), for all upper bounds \( w \) of \( \{y \land z_j\}_{j \in I} \). As \( y \land \sup_{j \in I} z_j \) is one of those upper bounds, it is the least upper bound of this set; that is,

\[
\sup_{j \in I} (y \land z_j) = y \land \sup_{j \in I} z_j.
\]

This proves that \( L \) is a semi-Heyting algebra.

It is clear that the dual tied adjointness algebra \( A^\text{dual} = (L^{op}, P, 1, A^d, K^d, H^d, T, I) \) (for which, see [15, Section 3.2]) is also sH-prelinear. So by the last conclusion above, \( L^{op} \) is a semi-Heyting algebra; that is, \( L \) is also a co-semi-Heyting algebra.

Finally, by applying our conclusions to the sH-prelinear tied adjointness algebra \((P, \leq_p, P, \leq_p, 1, I, T, I, T, I, \land, \lor, \neg, \top)\), we deduce that \( P \) too is a bi-semi-Heyting algebra. \( \square \)

We investigate the completeness of \( \text{L-AdjTV} \) with respect to semantics over sHL-ADJT. The following definition is adapted from [8] to suit \( \text{L-AdjTV} \):

**Definition 4.3.2.** Let \( \ell \) be a predicate language, and let \( A = (L, \leq_L, P, \leq_p, 1, A, K, H, T, I, \land, \lor, \neg, \top) \) be an sH-prelinear tied adjointness algebra. A \( \ell \)-interpretation (or \( \ell \)-structure) for the predicate language \( \ell \) is a structure

\[
\mathbb{M} = (M, (r_R)_{R \in \text{Pred}_P}, (r_Q)_{Q \in \text{Pred}_L}, (m_c)_{c \in \text{Const}}),
\]

where \( M \neq \emptyset, r_R : M^{ar(R)} \to P, r_Q : M^{ar(Q)} \to L \) and \( m_c \in M \) for each \( R \in \text{Pred}_P, Q \in \text{Pred}_L \) and \( c \in \text{Const} \). The two relations \( r_R \) and \( r_Q \) are fuzzy relations on \( M \) associating to each tuple \( (m_1, \ldots, m_{ar(R)}) \) of elements of \( M \) the degree \( r_R(m_1, \ldots, m_{ar(R)}) \in P \), and to each tuple \( (m_1, \ldots, m_{ar(Q)}) \) of elements of \( M \) the degree \( r_Q(m_1, \ldots, m_{ar(Q)}) \in L \).

An \( \mathbb{M} \)-evaluation of object variables is a mapping \( v : \text{Var} \to M \) assigning to each object variable \( x \) an element \( v(x) \in M \).

The value of a term given by \( \mathbb{M} \), \( v \) is defined as follows:

\[
\|x\|_{\mathbb{M}, v} = v(x) \in M, \\
\|c\|_{\mathbb{M}, v} = m_c \in M.
\]
The truth value $\|x\|_{M,v}^A$ of a formula $x$ in $WF_L$ and the truth value $\|\tau\|_{M,v}^A$ of a formula $\tau$ in $WF_P$ are defined inductively from

$\|R(t_1, \ldots , t_n)\|_{M,v}^A = r_R(\|t_1\|_{M,v}, \ldots, \|t_n\|_{M,v}) \in P$,

$\|Q(t_1, \ldots , t_n)\|_{M,v}^A = r_Q(\|t_1\|_{M,v}, \ldots, \|t_n\|_{M,v}) \in L$,

taking into account that the value commutes with connectives; by defining

$\|\zeta \Rightarrow \gamma\|_{M,v}^A = A(\|\zeta\|_{M,v}^A, \|\gamma\|_{M,v}^A) \in L$;

and so on for the connectives $\&$, $\lor$, $\ast$, $\rightarrow$, $\land$, and $\neg$.

Let $v, v'$ be two evaluations. $v \equiv_x v'$ means that $v(y) = v'(y)$ for each variable $y$ distinct from $x$.

$\|(\exists x)\phi\|_{M,v}^A = \sup\{\|\phi\|_{M,v'}^A | v \equiv_x v'\} \in L$;

$\|(\forall x)\phi\|_{M,v}^A = \inf\{\|\phi\|_{M,v'}^A | v \equiv_x v'\} \in L$;

$\|(\exists x)\tau\|_{M,v}^A = \sup\{\|\tau\|_{M,v'}^A | v \equiv_x v'\} \in P$;

$\|(\forall x)\tau\|_{M,v}^A = \inf\{\|\tau\|_{M,v'}^A | v \equiv_x v'\} \in P$.

The last line obviously means that $\|(\forall x)\tau\|_{M,v}^A = 1$ iff for all $v \equiv_x v'$, $\|\tau\|_{M,v'}^A = 1$; otherwise $\|(\forall x)\tau\|_{M,v}^A < 1$.

Observe that if $x$ is not free in $\phi$ then the value $\|\phi\|_{M,v}^A$ does not depend on $v(x)$, i.e. if $v \equiv_x v'$ then $\|\phi\|_{M,v}^A = \|\phi\|_{M,v'}^A$.

A truth-value becomes undefined if the infimum or supremum in its definition does not exist in $A$. A $A$-structure $M$ is said to be safe if all infima and suprema needed for defining the truth value of any formula exist in $A$.

**Definition 4.3.3.** (1) Let $\phi$ denote a formula in $WF_L \cup WF_P$ of a language $\mathcal{L}$ and let $\mathcal{M}$ be a safe $A$-interpretation for $\mathcal{L}$. The truth value of $\phi$ in $\mathcal{M}$ is

$$\|\phi\|_{M}^A = \inf\{\|\phi\|_{M,v}^A | v : Var \rightarrow M\}. \quad (137)$$

(2) A formula $\tau$ in $WF_P$ of a language $\mathcal{L}$ is said to be valid (or, true) in $\mathcal{M}$ if $\|\tau\|_{M}^A = 1$. $\tau$ is a $A$-tautology if $\|\tau\|_{M}^A = 1$ for each safe $A$-interpretation $\mathcal{M}$, i.e. $\|\tau\|_{M,v}^A = 1$ for each safe $A$-interpretation $\mathcal{M}$ and each $\mathcal{M}$-evaluation $v$ of object variables.

**Lemma 4.3.2.** The axioms $(\forall 1), (\exists 1), (\exists 2), (\exists 3), (\forall 4), (\exists 5), (\forall 5)$ and $(\exists 4)$ are $A$-tautologies for each prelinear tied adjointness algebra $A$, and the inference rules are sound for them. While the axiom $(\forall 3)$ is a $A$-tautology for each $sh$-prelinear tied adjointness algebra $A$.

**Proof.** The direct proof of the first assertion is a virtual replication of Hájek’s proofs of [8, Lemmas 5.1.9, 5.1.10].

For $(\forall 3)$, we only need to have $\inf_w(a \lor b_w) = a \lor \inf_w b_w$, which is the reason for requiring the lattice $P$ to be a co-semi-Heyting algebra. \(\square\)

**Definition 4.3.4.** Let $\Gamma$ be a theory over $L$-$AdjTV$, let $A$ be an $sh$-prelinear tied adjointness algebra and $\mathcal{M}$ a safe $A$-interpretation for the language of $\Gamma$. $\mathcal{M}$ is a $A$-model of $\Gamma$ if all axioms of $\Gamma$ are true in $\mathcal{M}$, i.e. $\|\zeta\|_{M}^A = 1$ for each $\zeta \in \Gamma$.

**Theorem 4.3.1 (Soundness).** Let $\Gamma$ be a theory over $L$-$AdjTV$ and $\tau$ be a formula in $WF_P$. If $\Gamma \vdash \tau$, then $\|\tau\|_{M}^A = 1$ for each $sh$-prelinear tied adjointness algebra $A$ and each $A$-model $\mathcal{M}$ of $\Gamma$.

**Proof.** This follows by the obvious induction on the length of a proof, using Lemma 4.3.2. \(\square\)

The assumption of the $sh$-prelinearity of $A$ (on $(L, P)$) proffers the following bonus besides soundness: since the meet $\neg\neg$ is the only conjunction assumed on $L$, we would want it to have a residuum. A necessary condition for that residuum to exist is that $L$ should be a semi-Heyting algebra, which is guaranteed by Lemma 4.3.1 when $A$ is $sh$-prelinear. This condition becomes also sufficient when $L$ is complete.

In proving the $shL$-$ADJT$ and chain completeness of $L$-$AdjTV$, we follow the same methods adopted by Hájek [8] within $BL\forall$. 


Definition 4.3.5 ( Hájek [8]). A theory \( \Gamma \) over \( \textbf{L-AdjTV} \) is called Henkin if for each closed formula in \( WFP \) of the form \( (\forall x) \zeta(x) \) unprovable in \( \Gamma \) there is a constant \( c \) in the language of \( \Gamma \) such that \( \zeta(c) \) is unprovable in \( \Gamma \).

Let \( A_\Gamma \) be the prelinear tied adjointness algebra of classes of \( \Gamma \)-equivalid closed formulae; defined exactly as in Definition 3.4.1. Recall that the top element \( 1_\Gamma \) of \( WFP/\sim_p \) is the \( \Gamma \)-equivalidity class of all closed formulae \( \zeta \) in \( WFP \) such that \( \Gamma \vdash \zeta \).

Lemma 4.3.3. If \( \Gamma \) is Henkin, then in \( A_\Gamma \) we find for all \( x \) in \( WF_L \) and \( \zeta \) in \( WFP \) with just one free variable \( x \) that (\( c \) runs over all object constants of \( \Gamma \)):

\[
[(\forall x) \zeta]^P = \inf_c [(\exists x) \zeta]^P, \tag{138}
\]

\[
[(\exists x) \zeta]^P = \sup_c [(\forall x) \zeta]^P, \tag{139}
\]

\[
[(\forall x) \zeta]_1^P = \inf_c [(\exists x) \zeta]_1^P \tag{140},
\]

\[
[(\exists x) \zeta]^P = \sup_c [(\forall x) \zeta]^P \tag{141}.
\]

A \( \Gamma \)-model \( \mathcal{M}_\Gamma \) of \( \Gamma \) is obtained by taking \( M \) to be the set of all constants of the language of \( \Gamma \); \( m_c = c \) for each such constant, and for each predicate \( R \in \text{Pred}_p \), we take \( r_R(c_1, \ldots, c_n) = [R(c_1, \ldots, c_n)]^P \), and for each predicate \( Q \in \text{Pred}_L \), \( r_Q(c_1, \ldots, c_n) = [Q(c_1, \ldots, c_n)]^P \).

Proof (Cf. Hájek [8]). By axiom (\( \forall x \)), \( [(\forall x) \zeta(x)]_1^P \leq L [(\exists x) \zeta(x)]_1^P \) for each \( c \). Thus \( [(\forall x) \zeta(x)]_1^P \leq \inf_c [(\exists x) \zeta(x)]_1^P \). To prove \( [(\forall x) \zeta(x)]_1^P \leq \inf_{c} [(\exists x) \zeta(x)]_1^P \) for each \( c \), we have to prove \( [(\forall x) \zeta(x)]_1^P \leq [L [(\exists x) \zeta(x)]_1^P \].

But if \( [(\exists x) \zeta(x)]_1^P \leq L [(\forall x) \zeta(x)]_1^P \) then \( \forall \gamma \vdash [(\forall x) \zeta(x)]_1^P \). Thus by Theorem 4.2.3, \( \forall \gamma \vdash [(\forall x) \zeta(x)]_1^P \). Hence, by the Henkin property, there is a constant \( c \) such that \( \forall \gamma \vdash \zeta(c) \), thus \( (\forall x) \zeta(c) \vdash [L [(\exists x) \zeta(x)]_1^P \), a contradiction. This proves (138).

Similarly, \( [(\exists x) \zeta(x)]_1^P \leq \inf_{c} [(\forall x) \zeta(x)]_1^P \) for each \( c \), by axiom (\( \exists x \)). Assume \( [(\exists x) \zeta(x)]_1^P \leq [L [(\forall x) \zeta(x)]_1^P \) for each \( c \); we have to prove \( [(\exists x) \zeta(x)]_1^P \leq L [(\forall x) \zeta(x)]_1^P \). Indeed, if \( [(\exists x) \zeta(x)]_1^P \leq L [(\forall x) \zeta(x)]_1^P \) then \( \forall \gamma \vdash [(\exists x) \zeta(x)]_1^P \). Thus by Theorem 4.2.3, \( \forall \gamma \vdash [(\exists x) \zeta(x)]_1^P \). Hence, by the Henkin property, there is a constant \( c \) such that \( \forall \gamma \vdash \zeta(c) \), thus \( (\exists x) \zeta(c) \vdash [L [(\forall x) \zeta(x)]_1^P \), a contradiction, which yields (139).

Eqs. (140) and (141) are given very similar proofs in [8]. It remains to prove \( \|\zeta\|_{M_\Gamma}^{A_\Gamma} = [\zeta]_1^P \) and \( \|\zeta\|_{M_\Gamma}^{A_\Gamma} = [(\exists x) \zeta(x)]_1^P \) for all closed formalu in \( WFP \) and \( x \) in \( WF_L \) (we simply write \( M \) for \( M_\Gamma \)). Then for each axion \( \tau \) of \( \Gamma \) we have \( \|\tau\|_{M_\Pi}^{A_\Pi} = [1]_1^P \). We use complete induction on the complexity of closed formalu. For atomic closed formalu the claim follows by definition, and the induction step for connectives is obvious. We handle the quantifiers. Let \( (\forall x) \zeta \), \( (\exists x) \zeta \) be closed formalu in \( WF_L \). Then, by the induction hypothesis and (138), (139), we get

\[
\|[(\forall x) \zeta(x)]\|_{M_\Gamma}^{A_\Gamma} = \inf_c \|\zeta(c)\|_{M_\Gamma}^{A_\Gamma} = \inf_c [(\exists x) \zeta(x)]_1^P = [(\forall x) \zeta(x)]_1^P \quad \text{and}
\]

\[
\|[(\exists x) \zeta(x)]\|_{M_\Gamma}^{A_\Gamma} = \sup_c \|\zeta(c)\|_{M_\Gamma}^{A_\Gamma} = \sup_c [(\forall x) \zeta(x)]_1^P = [(\exists x) \zeta(x)]_1^P.
\]

and similarly for closed formalu in \( WFP \). This completes the proof by induction.

Definition 4.3.6 ( Hájek [8]). A theory \( \Gamma \) over \( \textbf{L-AdjTV} \) is said to be linearly complete (it is called complete in [8]) if for each pair \( \zeta, \tau \) of closed formalu in \( WFP \),

\( \Gamma \vdash \zeta \rightarrow \tau \) or \( \Gamma \vdash \tau \rightarrow \zeta \).

Lemma 4.3.4. Let \( \Gamma \) be a theory over \( \textbf{L-AdjTV} \). Then

(i) \( \Gamma \) is linearly complete iff for each pair \( \zeta, \tau \) of closed formalu in \( WFP \) such that \( \Gamma \vdash \zeta \lor \tau \), \( \Gamma \) proves \( \zeta \) or \( \Gamma \) proves \( \tau \) [8].

(ii) If \( \Gamma \) is a linearly complete theory then for each pair \( \zeta, \beta \) of closed formalu in \( WF_L \): \( \Gamma \vdash (\zeta \supset \beta) \) or \( \Gamma \vdash (\beta \supset \zeta) \).

(iii) \( \Gamma \) is linearly complete if and only if \( A_\Gamma \) is linearly ordered.
Theorem 4.3.2 (Cf. Hájek [8]). Let \( \Gamma \) be a theory over \( \text{L-AdjTF} \) and let \( \xi, \tau \) be closed formulae in \( \text{WF}_P \) of the language of \( \Gamma \). Then \( \Gamma \cup \{ \xi \} \vdash \tau \) iff there is an \( n \) such that \( \Gamma \vdash \xi^n \rightarrow \tau \) (where \( \xi^n = \xi \ast \ldots \ast \xi \), \( n \) factors).

Lemma 4.3.5 (Cf. Hájek [8]). For each theory \( \Gamma \) over \( \text{L-AdjTF} \) and all closed formulae \( \xi, \chi, \eta \) in \( \text{WF}_P \), we have

(i) If \( \Gamma \vdash \chi \lor \eta \) and \( \Gamma \nvdash \xi \), then either \( \Gamma \cup \{ \chi \} \nvdash \xi \) or \( \Gamma \cup \{ \eta \} \nvdash \xi \).
(ii) If \( \Gamma \nvdash \xi \), then for each pair \( (\xi, \tau) \) of closed formulae in \( \text{WF}_P \), either \( \Gamma \cup \{ \xi \rightarrow \tau \} \nvdash \xi \) or \( \Gamma \cup \{ \tau \rightarrow \xi \} \nvdash \xi \), and for each pair \( (\alpha, \beta) \) of closed formulae in \( \text{WF}_L \), either \( \Gamma \cup \{ \alpha \supset \beta \} \nvdash \xi \) or \( \Gamma \cup \{ \beta \supset \alpha \} \nvdash \xi \).
(iii) If \( \Gamma \vdash \chi \lor \eta \), then \( \Gamma \cup \{ \eta \rightarrow \chi \} \vdash \chi \) (closedness is not needed).

Proof. (Cf. Hájek [8]). (i) Suppose \( \Gamma \cup \{ \chi \} \vdash \xi \) and \( \Gamma \cup \{ \eta \} \vdash \xi \). Then by the deduction theorem (Theorem 4.3.2), for some \( k \), \( \Gamma \vdash \chi^k \rightarrow \xi \) and \( \Gamma \vdash \eta^k \rightarrow \xi \). Therefore, by Proposition 3.3.3, \( \Gamma \vdash \chi^k \lor \eta^k \rightarrow \xi \). We use prelinearity, through Corollary 3.2.1: \( \chi \lor \eta \vdash \chi^k \lor \eta^k \). Hence \( \Gamma \vdash \xi \), a contradiction.
(ii) Follows from (i) and the two prelinearity Eqs. (67) and (96).
(iii) This follows from the equivalidity \( (\eta \rightarrow \chi) \leftrightarrow (\chi \lor \eta \rightarrow \chi) \) (59).

Lemma 4.3.6 (Cf. Hájek [8]). For each theory \( \Gamma \) over \( \text{L-AdjTF} \) and each closed formula \( \xi \) in \( \text{WF}_P \), if \( \Gamma \nvdash \xi \) then there is a linearly complete Henkin theory \( \hat{\Gamma} \supseteq \Gamma \) such that \( \hat{\Gamma} \nvdash \xi \).

Proof. The proof uses Lemma 4.3.5, and is an exact repetition of the proof within BL provided by Hájek for [8, Lemma 5.2.7]. We note that prelinearity and the axiom (\( \forall \exists \)) are employed in the proof.

Theorem 4.3.3 (Completeness). Let \( \Gamma \) be a theory over \( \text{L-AdjTF} \) and let \( \xi \) be a formula in \( \text{WF}_P \) of the language of \( \Gamma \). Then the following three statements are equivalent:

(i) \( \Gamma \) proves \( \xi \).
(ii) For each \( \text{H}-\text{prelinear} \) tied adjointness algebra \( A \) and each \( A \text{-model} \) \( \mathcal{M} \) of \( \Gamma \), \( \| \xi \|_{\mathcal{M}}^A = 1 \).
(iii) For each tied adjointness chain \( A \) and each \( A \text{-model} \) \( \mathcal{M} \) of \( \Gamma \), \( \| \xi \|_{\mathcal{M}}^A = 1 \).

Proof. (i) entails (ii): This is the Soundness Theorem.
(ii) entails (iii): Trivial.
(iii) entails (i): Suppose \( \Gamma \nvdash \xi \), and assume, without loss of generality, that \( \xi \) has one free variable \( x \). Then \( \Gamma \nvdash (\forall x) \xi(x) \). Then by Lemma 4.3.6 there is a linearly complete Henkin supertheory \( \hat{\Gamma} \) of \( \Gamma \) such that \( \hat{\Gamma} \nvdash (\forall x) \xi(x) \); i.e., \( \| \xi \|_{\mathcal{M}_{\hat{\Gamma}}}^{A_{\hat{\Gamma}}} = \| (\forall x) \xi(x) \|_{\mathcal{M}_{\hat{\Gamma}}}^{A_{\hat{\Gamma}}} \neq 1_{\hat{\Gamma}} \) in the \( A_{\hat{\Gamma}} \)-model \( \mathcal{M}_{\hat{\Gamma}} \) of \( \hat{\Gamma} \), constructed in Lemma 4.3.3. The linearity of \( A_{\hat{\Gamma}} \) is guaranteed by Lemma 4.3.4(iii). As \( \hat{\Gamma} \supseteq \Gamma \), \( \mathcal{M}_{\hat{\Gamma}} \) is also a \( A_{\hat{\Gamma}} \)-model of \( \Gamma \), and we are done.

5. Conclusion

We showed that a Hilbert system, based on seven axioms and one deduction rule (modus ponens), is complete for the propositional calculus \text{AdjTPC} of tied adjointness algebras.

We then turned to a syntactical study of prelinear tied adjointness algebras, in which two comparators \( (H \text{ and } I) \) were prelinear. We constructed propositional and predicate calculi for them, and we established two types of completeness for each calculus.

We managed to reduce the number of proofs in each calculus, through an adaptation of a duality principle introduced by Morsi [14].
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References